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STUDY OF A COLLECTION OF SERVICE TIME DISTRIBUTIONS AND IMPACT IN THE RESPECTIVE M|G|∞ SYSTEM BUSY PERIOD PARAMETERS

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Abstract. The problems arising when computing the moments of a particular service time distributions collection, for which the M|G|∞ queue system busy period becomes very easy to study, are presented and it is shown how to overcome them. Some results, precisely about the moment's computation of random variables with distribution functions given by this collection are given. The busy period "peakedness" and "modified peakedness" for the M|G|∞ queue in the case of those service time distributions are also computed.

Keywords: service time, collection, distributions, moments, M|G|∞ queue, busy period

Mathematics Subject Classification: 60K35

1 Introduction

If, in the M|G|∞ queue system, the service time length is a random variable with a distribution function belonging to the collection

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (1.1),$$

the busy period length probability distribution is exponential with an atom at the origin, see Ferreira (2005) and (Ferreira and Andrade, 2009). But although it is so easy to study the busy period in this situation it is very difficult to compute the service time moments.

Some results, precisely about the moment's computation of random variables with distribution functions given by this collection are given.

In the end are presented formulae that give the "peakedness" and the "modified peakedness" to the busy period of the M|G|∞ system for those service time distributions, see Ferreira (2004, 2013,2013a).

This work is built on a part of the presented in Ferreira (2007) which is so corrected, generalized and updated.

2 Moments Calculation Problems

Be $G(t), t \geq 0$ a distribution function and $g(t) = \frac{dG(t)}{dt}$. The differential equation $(1-p)\frac{g(t)}{1-G(t)} - \lambda p - \lambda(1-p)G(t) = \beta$, where $\lambda > 0$ and $-\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}$, $0 \leq p < 1$ ($\rho = \lambda\alpha$, being α the mean of $G(t)$) has (1.1) as solution (see Ferreira (2005)). If, in (1.1), $G_i(t)$ is the solution associated to ρ_i, $i = 1,2,3,4$ it is easy to see that

$$\frac{G_4(t)-G_2(t)}{G_4(t)-G_1(t)} \cdot \frac{G_3(t)-G_1(t)}{G_3(t)-G_2(t)} = \frac{e^{-\rho_4}-e^{-\rho_2}}{e^{-\rho_4}-e^{-\rho_1}} \cdot \frac{e^{-\rho_3}-e^{-\rho_1}}{e^{-\rho_3}-e^{-\rho_2}} \qquad (2.1)$$

as it had to happen since the differential equation dealt with is a Riccati equation. And computing,

$$\int_0^{\infty}[1-G(t)]dt = \int_0^{\infty} \frac{\left(1-e^{-\rho}\right)\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda} dt =$$

$$= \frac{\left(1-e^{-\rho}\right)\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda} \int_0^{\infty} \frac{1}{e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+1} dt =$$

$$= \frac{\left(1-e^{-\rho}\right)\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda} \int_0^{\infty} \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}-e^{-\rho}e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}+e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}} dt =$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\int_0^\infty \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}dt=$$

$$=\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda}\cdot\frac{-1}{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}\cdot\left[\log\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\right]_0^\infty=$$

$$=-\frac{1}{\lambda}\left(\log e^{-\rho}-\log 1\right)=\frac{-\rho}{-\lambda}=\alpha$$

as it had to be because are considered positive random variable. The density associated to $G(t)$ given by (1.1) is

$$g(t)=\frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2 e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\lambda\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2},t>0,-\lambda\le\beta\le\frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1},0\le p<1 \quad (2.2).$$

So,

$$\int_0^\infty t^n g(t)dt=\frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2}{\lambda}\cdot\int_0^\infty t^n\frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\cdot$$

But, $\int_0^\infty t^n\frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt\ge\int_0^\infty t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt=$

$$=\frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}}\frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n},\beta\ne-\lambda$$. And,

$$\int_0^{\infty} t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}dt \le e^{2\rho}\int_0^{\infty} t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}dt =$$

$$= \frac{e^{2\rho}}{\lambda+\frac{\lambda p+\beta}{1-p}} \frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, \beta \ne -\lambda .$$

So, calling T the random variable corresponding to $G(t)$:

$$\frac{\left(1-e^{-\rho}\right)e^{-\rho}}{\lambda} \frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}} \le \mathrm{E}\left[\mathrm{T}^n\right] \le \frac{e^{\rho}-1}{\lambda} \frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n-1}},$$

$$,-\lambda < \beta \le \frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1}, 0 \le p < 1, n = 1,2,... \qquad (2.3).$$

Notes:

- The expression (2.3), giving bounds for $\mathrm{E}\left[\mathrm{T}^n\right]$, guarantees its existence,
- For $n=1$ the expression (2.3) is useless since $\mathrm{E}[\mathrm{T}]=\alpha$. Note, curiously, that here the upper bound is $\frac{e^{\rho}-1}{\lambda}$, the M|G|∞ system busy period mean value, evidently disparate,
- For $n=2$, subtracting to both bounds α^2, it is possible get from expression (2.3) bounds for $VAR[\mathrm{T}]$,
- For $\beta=-\lambda, \mathrm{E}\left[\mathrm{T}^n\right]=0, n=1,2,...,$ evidently.

See, however, that (1.1) can be written like:

$$G(t)=\frac{1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{1-\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}, t \ge 0, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \qquad (2.4)$$

and, for $\rho < \log 2$,

$$G(t)=\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\cdot\sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^{k}e^{-k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t},$$

$$,t\geq 0,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (2.5).$$

After (2.5) it is easy to derive the T Laplace Transform for $\rho<\log 2$. And, so,

- For $\rho<\log 2$

$$\mathrm{E}\left[\mathrm{T}^{n}\right]=-\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right)n!\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^{n}},-\lambda<\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1,$$

$$,n=1,2,... \qquad (2.6).$$

Notes:

- $\mathrm{E}[\mathrm{T}]=-\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right)\sum_{k=1}^{\infty}\frac{\left(1-e^{\rho}\right)^{k}}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}=\frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(1-e^{\rho}\right)}{k}=$

$$=\frac{1}{\lambda}\log e^{\rho}=\frac{\rho}{\lambda}=\alpha.$$

- For $n\geq 2$ only a finite number of parcels can be considered in the infinite sum. Calling M this number, to get an error lesser than ε it must be fulfilled simultaneously

a) $\mathrm{M}>\frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}}-1,$

b) $\mathrm{M}>\log_{\left(e^{\rho}-1\right)}\frac{\varepsilon e^{\rho}\lambda}{n!\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}-1.$

So it is evident now that this distributions collection moment's computation is a complex task. This was already true for the study of Ferreira (1998) where the results presented are a particular situation of these ones for $p = 0$. To consider the approximation

$$\mathrm{E}_m^n = \sum_{k=1}^{\infty} \left(\frac{k}{m}\right)^n \left[G\left(\frac{k}{m}\right) - G\left(\frac{k-1}{m}\right)\right], -\lambda < \beta \leq \frac{\lambda\left(1 - pe^{\rho}\right)}{e^{\rho} - 1}, 0 \leq p < 1, n = 1,2,... \qquad (2.7)$$

may be helpful since $\lim_{m\to\infty} \mathrm{E}_m^n = \mathrm{E}\left[\mathrm{T}^n\right], n = 1,2,...$ (Ferreira and Andrade, 2012c) that allow the moments numerical computation .

3 The "Peakedness" and the "Modified Peakedness" for the M|G|∞ Queue Busy Period

The M|G|∞ queue busy period "peakedness" is the Laplace transform of its length at $\frac{1}{\alpha}$, Ferreira (2013,2013a) . It is a parameter that characterizes the busy period distribution length and contains information about all its moments. For the collection of service distributions (1.1) the "peakedness", named pi , is

$$pi = \frac{e^{-\rho}\left(\lambda + \beta\right)\left(\rho + 1\right) - \lambda p - \beta}{\lambda\left(e^{-\rho}\left(\rho + \alpha\beta\right) + 1 - p\right)}, -\lambda \leq \beta \leq \frac{\lambda\left(1 - pe^{\rho}\right)}{e^{\rho} - 1}, 0 \leq p < 1 \qquad (3.1).$$

In Ferreira (2013,2013a) is also introduced another measure, the "modified peakedness" got after the "peakedness" taking out the terms that are permanent for the busy period in different service distributions and putting over the common part. Calling it qi:

$$qi = pi\frac{\rho}{e^{\rho} - \rho - 1} + 1$$

and so, for the distributions given by collection (1.1):

$$qi = \frac{e^{-\rho}\left(\lambda + \beta\right)\left(\rho + 1\right) - \lambda p + \beta}{\lambda\left(e^{-\rho}\left(\rho + \alpha\beta\right) + 1 - p\right)}\frac{\rho}{e^{\rho} - \rho - 1} + 1, -\lambda \leq \beta \leq \frac{\lambda\left(1 - pe^{\rho}\right)}{e^{\rho} - 1},\ \ 0 \leq p < 1 \ \ (3.2).$$

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