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ON L-PACKETS AND DEPTH FOR $\mathbf{SL_2}(K)$ AND ITS INNER FORM

Anne-Marie Aubert
Institut de Mathématiques de Jussieu, Paris Rive Gauche, U.M.R. 7586 du C.N.R.S., U.P.M.C., 4 place Jussieu 75005 Paris, France
anne-marie.aubert@img-prg.fr

Sergio Mendes
ISCTE - Lisbon University Institute, Av. das Forças Armadas, 1649-026, Lisbon, Portugal
sergio.mendes@iscte.pt

Roger Plymen
School of Mathematics, Southampton University, Southampton SO17 1BJ, UK
r.j.plymen@soton.ac.uk

Maarten Solleveld
IMAPP, Radboud Universiteit, Heyendaalseweg 135, 6525AJ, Nijmegen, the Netherlands
m.solleveld@science.ru.nl

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We consider the group $\mathrm{SL}_2(K)$, where K is a local non-archimedean field of characteristic two. We prove that the depth of any irreducible representation of $\mathrm{SL}_2(K)$ is larger than the depth of the corresponding Langlands parameter, with equality if and only if the L-parameter is essentially tame. We also work out a classification of all L-packets for $\mathrm{SL}_2(K)$ and for its non-split inner form, and we provide explicit formulae for the depths of their L-parameters.

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1. Introduction

Let K be a non-archimedean local field and let K_{s} be a separable closure of K. A central role in the representation theory of reductive K-groups is played by the local Langlands correspondence (LLC). It is known to exist in particular for the inner forms of the groups $\mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$, and to preserve interesting arithmetic information, like local L-functions and ϵ-factors.

Another invariant that makes sense on both sides of the LLC is *depth*. The *depth* $d(\pi)$ of an irreducible smooth representation π of a reductive p-adic group $\mathcal{G}$ was defined by Moy and Prasad [13] in terms of filtrations $\mathcal{G}_{x,r}$ ($r \in \mathbb{R}_{\geq 0}$) of its parahoric subgroups $\mathcal{G}_x$. The depth of a Langlands parameter ϕ is defined to be the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(F_s/F)^r$ for all $r > d(\phi)$, where $\mathrm{Gal}(K_{\mathrm{s}}/K)^r$ is the r-th ramification subgroup of the absolute Galois group of K.

Let D be a division algebra with centre K, of dimension d^2 over K. Then $\mathrm{GL}_m(D)$ is an inner form of $\mathrm{GL}_n(K)$ with $n = dm$. There is a reduced norm map Nrd: $\mathrm{GL}_m(D) \to K^\times$ and the derived group $\mathrm{SL}_m(D) := \ker(\mathrm{Nrd}\colon \mathcal{G} \to K^\times)$ is an inner form of $\mathrm{SL}_n(K)$. Every inner form of $\mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$ is isomorphic to one of this kind. When $n = 2$, the only possibilities for d are 1 or 2, and so the inner forms are, up to isomorphism, $\mathrm{GL}_2(K)$ and $D^\times$, and $\mathrm{SL}_2(K)$ and $\mathrm{SL}_1(D)$.

The LLC for $\mathrm{GL}_m(D)$ preserves the depth, that is, for every smooth irreducible representation π of $\mathrm{GL}_m(D)$, we have $d(\pi) = d(\varphi_\pi)$, where φ_π corresponds to π by the LLC [1, Theorem 2.9].

The situation is different for $\mathrm{SL}_m(D)$. All the irreducible representations in a given L-packet Π_ϕ have the same depth, so the depth is an invariant of the L-packet, say $d(\Pi_\phi)$. We have $d(\Pi_\phi) = d(\varphi)$ where φ is a lift of ϕ which has minimal depth among the lifts of ϕ, and the following holds:

$$d(\phi) \leq d(\Pi_\phi) \tag{1.1}$$

for any Langlands parameter ϕ for $\mathrm{SL}_m(D)$ [1, Proposition 3.4 and Corollary 3.4]. Moreover (1.1) is an equality if ϕ is *essentially tame*, that is, if the image by ϕ of the wild inertia subgroup $\mathbf{P}_K$ of the Weil group $\mathbf{W}_K$ of K lies in a maximal torus of $\mathrm{PGL}_n(\mathbb{C})$.

We observe that this notion of essentially tameness is consistent with the usual notion for Langlands parameters for $\mathrm{GL}_n(K)$. Indeed, any lift $\varphi\colon \mathbf{W}_K \to \mathrm{GL}_n(\mathbb{C})$ of ϕ, is called essentially tame if its restriction to $\mathbf{P}_K$ is a direct sum of characters. Clearly φ is essentially tame if and only if $\varphi(\mathbf{P}_K)$ lies in a maximal torus of $\mathrm{GL}_n(\mathbb{C})$, which in turn is equivalent to $\phi(\mathbf{P}_K)$ lying in a maximal torus of $\mathrm{PGL}_n(\mathbb{C})$.

We denote by $t(\varphi)$ the *torsion number* of φ, that is, the number of unramified characters χ of $\mathbf{W}_K$ such $\varphi\chi \cong \varphi$. Then ϕ and φ are essentially tame if and only if the residual characteristic p of K does not divide $n/t(\varphi)$ [4, Appendix].

In this article we take K to be a local non-archimedean field K of characteristic 2. In positive characteristic, K is of the form $K = \mathbb{F}_q((t))$, the field of Laurent series with coefficients in $\mathbb{F}_q$, with $q = 2^f$. This case is particularly interesting because there are countably many quadratic extensions of $\mathbb{F}_q((t))$. These quadratic extensions are parametrised by the cosets in $K/\wp(K)$ where $\wp$ is the map, familiar from Artin-Schreier theory, given by $\wp(X) = X^2 - X$.

We first show that equality holds in (1.1) only if ϕ is essentially tame (*i.e.*, $t(\varphi) = 2$):

Theorem 1.1. *Let K be a non-archimedean local field of characteristic 2, and let π be an irreducible representation of an inner form of $\mathrm{SL}_2(K)$, with Langlands parameter ϕ. If ϕ is not essentially tame then we have*

$$d(\pi) > d(\phi).$$

Let φ be a lift of ϕ with minimal depth among the lifts of ϕ. In the proof we distinguish the cases where φ is imprimitive, respectively primitive.

An irreducible Langlands parameter $\varphi\colon \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$ is called *imprimitive* if there exists a separable quadratic extension L of K and a character ξ of $L^\times$ such that $\varphi \simeq \mathrm{ind}_{\mathbf{W}_L}^{\mathbf{W}_K}(\xi)$. Then the depth of φ and ϕ may be expressed in terms of that of ξ and ξ^2, respectively, as

$$d(\varphi) = \big(d(\xi) + d(L/K)\big)/2 \quad \text{and} \quad d(\phi) = \big(d(\xi^2) + d(L/K)\big)/2,$$

where $\mathfrak{p}_K^{1+d(L/K)}$ is the relative discriminant of L/K. Let $\mathfrak{T}(\varphi)$ be the group of characters χ of $\mathbf{W}_K$ such that $\chi \otimes \varphi \simeq \varphi$. As in [3, 41.4], we call φ *totally ramified* if $\mathfrak{T}(\phi)$ does not contain any unramified character. If φ is not essentially tame, then it is totally ramified. We check that if this case we have $d(\xi) > d(\xi^2)$, and hence $d(\Pi_\phi) > d(\phi)$.

We obtain in Proposition 3.2 the following characterization of L-packets for $\mathrm{SL}_2(K)$ or $\mathrm{SL}_1(D)$: an L-packet is a minimal set of irreducible representations from which a stable distribution can be constructed.

Next we give the explicit classification of the L-packets for both $\mathrm{SL}_2(K)$ and $\mathrm{SL}_1(D)$.

In particular, to each biquadratic extension L/K, there is attached a Langlands parameter $\phi = \phi_{L/K}$, and an L-packet Π_ϕ of cardinality 4. The depth of the parameter $\phi_{L/K}$ depends on the extension L/K. More precisely, the numbers $d(\phi)$ depend on the breaks in the upper ramification filtration of the Galois group $\mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let D be a central division algebra of dimension 4 over K. The parameter ϕ is relevant for the inner form $\mathrm{SL}_1(D)$, which admits singleton L-packets.

Theorem 1.2. *Let L/K be a biquadratic extension, let ϕ be the Langlands parameter $\phi_{L/K}$. If the highest break in the upper ramification of the Galois group $\mathrm{Gal}(L/K)$ is t then we have $d(\phi) = t$. For every $\pi \in \Pi_\phi(\mathrm{SL}_2(K)) \cup \Pi_\phi(\mathrm{SL}_1(D))$ these integers provide lower bounds:*

$$d(\pi) \geq d(\phi).$$

Depending on the extension L/K, all the odd numbers $1, 3, 5, 7, \ldots$ are achieved as such breaks.

This contrasts strikingly with the case of $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a unique biquadratic extension L/K, and a unique tamely ramified discrete parameter $\phi : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$ of depth zero.

Let E/K be the quadratic extension given by

$$E = K(\wp^{-1}(\varpi^{-2n-1}))$$

with ϖ a uniformizer and $n = 0, 1, 2, 3, \ldots$ and let ϕ_E be the associated L-parameter. We prove in Subsection 3.4 that the depth of ϕ_E is given by

$$d(\phi_E) = 2n + 1.$$

For the L-packets considered in this article, the depths $d(\pi)$ can be arbitrarily large.

Section 4 is devoted to aspects of the Artin-Schreier theory. This section goes a little further than the exposition in [8, p.146–151] and the article of Dalawat [6]. We have occasion to refer to this section at several points in our article.

We thank Chandan Dalawat for a valuable exchange of emails and for bringing the reference [6] to our attention.

2. Depth of *L*-parameters

The field K possesses a central division algebra D of dimension 4 and, up to isomorphism, only one. The group $D^\times$ is locally profinite and is compact modulo its centre $K^\times$, see [3, p.325]. Let Nrd denote the reduced norm on $D^\times$. Define

$$\mathrm{SL}_1(D) = \{x \in D^\times : \mathrm{Nrd}(x) = 1\}.$$

Then $\mathrm{SL}_1(D)$ is an inner form of $\mathrm{SL}_2(K)$. The articles [11,2] finalize the local Langlands correspondence for any inner form of SL_n over all local fields.

Depth of an L-parameter for $\mathrm{GL}_2(K)$. Let $\mathbf{W}_K$ denote the Weil group of K, and let $\Phi(\mathrm{GL}_2(K))$ be the set of L-parameters $\varphi\colon \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ for inner forms of

$\mathrm{GL}_2(K)$. Let t be a real number, $t \geq 0$, let $\mathrm{Gal}(K_{\mathrm{s}}/K)^t$ be the t-th ramification subgroup of the absolute Galois group of K. We define

$$\Phi_t(\mathrm{GL}_2(K)) := \{\varphi \in \Phi(\mathrm{GL}_2(K)) \,:\, \mathrm{Gal}(K_{\mathrm{s}}/K)^t \subset \ker(\varphi)\}. \tag{2.1}$$

Notice that $\Phi_{t'}(\mathrm{GL}_2(K)) \subset \Phi_t(\mathrm{GL}_2(K))$, if $t' \leq t$. It is known that the set of t's at which $\mathrm{Gal}(F_s/F)^t$ breaks consists of rational numbers and is discrete [14, Chap. IV, §3]. In particular there exists a unique rational number $d(\varphi)$, called the *depth* of φ, such that

$$\varphi \notin \Phi_{d(\varphi)}(\mathrm{GL}_2(K)) \quad \text{and} \quad \varphi \in \Phi_t(\mathrm{GL}_2(K)) \text{ for any } t > d(\varphi). \tag{2.2}$$

Depth of an L-parameter for $\mathrm{SL}_2(K)$. The depth of an L-parameter $\phi\colon \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ for an inner form of $\mathrm{SL}_2(K)$ is defined as:

$$d(\phi) = \inf\{t \in \mathbb{R}_{\geq 0} \mid \mathrm{Gal}(K_{\mathrm{s}}/K)^{t+} \subset \ker \phi\}, \tag{2.3}$$

where

$$\mathrm{Gal}(K_{\mathrm{s}}/K)^{t+} := \bigcap_{r>t} G^r.$$

Each projective representation $\phi\colon \mathbf{W}_K \to \mathrm{PGL}_2(\mathbb{C})$ lifts to a Galois representation

$$\varphi\colon \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C}).$$

For any such lift φ of ϕ we have $\ker(\varphi) \subset \ker \phi$, so

$$d(\varphi) \geq d(\phi). \tag{2.4}$$

Let $\varphi\colon \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$ be a 2-dimensional irreducible representation of $\mathbf{W}_K$, and let $\mathfrak{T}(\varphi)$ be the group of characters χ of $\mathbf{W}_K$ such that $\chi \otimes \varphi \simeq \varphi$. Then φ is primitive if $\mathfrak{T}(\varphi) = \{1\}$, *simply imprimitive* if $\mathfrak{T}(\varphi)$ has order 2, and *triply imprimitive* if $\mathfrak{T}(\varphi)$ has order 4, as in [3, 41.3]. Comparing determinants, we see that every nontrivial element of $\mathfrak{T}(\varphi)$ has order 2.

As in [3, 41.4], we call ϕ and φ *unramified* if $\mathfrak{T}(\varphi) \setminus \{1\}$ contains an unramified character, and *totally ramified* if $\mathfrak{T}(\varphi) \setminus \{1\}$ does not contain any unramified character. By definition, a primitive representation is totally ramified. Thus every imprimitive irreducible representation of dimension 2 of $\mathbf{W}_K$ which is not totally ramified is essentially tame.

Let $\phi\colon \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ with trivial restriction to $\mathrm{SL}_2(\mathbb{C})$, and such that φ is a lift of ϕ. If φ is essentially tame and has minimal depth among the lifts of ϕ, then we have $d(\phi) = d(\varphi)$ [1, Theorem 3.8]. Thus we are reduced to computing the depths of the projective representations of $\mathbf{W}_K$ which lift to totally ramified representations.

We recall how the depth of an irreducible representation (φ, V) of $\mathbf{W}_K$ can be computed. Put $E = (K_{\mathrm{s}})^{\ker \varphi}$, so that ϕ factors through $\mathrm{Gal}(E/K)$. Let g_j be the order of the ramification subgroup $\mathrm{Gal}(E/K)_j$ (in the lower numbering). The Artin conductor $a(\varphi) = a(V)$ is given by

$$a(\varphi) = g_0^{-1} \sum_{j \geq 0} g_j \dim\bigl(V/V^{\mathrm{Gal}(E/K)_j}\bigr) \in \mathbb{Z}_{\geq 0}. \tag{2.5}$$

Since (φ, V) is irreducible and $\mathrm{Gal}(E/K)_j$ is normal in $\mathrm{Gal}(E/K)$, $V^{\mathrm{Gal}(E/K)_j} = 0$ whenever $g_j > 1$. Thus (2.5) simplifies to the formula [10, (1)]:

$$a(\varphi) = \frac{\mathrm{dim} V}{g_0} \sum_{j \geq 0 : g_j > 1} g_j = \mathrm{dim} V + \frac{\mathrm{dim} V}{g_0} \sum_{j \geq 1 : g_j > 1} g_j \tag{2.6}$$

It was shown in [2, Lemma 4.1] that

$$d(\varphi) := \begin{cases} 0 & \text{if } \mathbf{I}_F \subset \ker(\phi), \\ \dfrac{a(\varphi)}{\dim V} - 1 & \text{otherwise.} \end{cases} \tag{2.7}$$

Let $\varphi\colon \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$ be a totally ramified irreducible representation. Let $\phi\colon \mathbf{W}_K \to \mathrm{PGL}_2(\mathbb{C})$ be its projection. We will show that $d(\varphi) > d(\phi)$. To this end we may and will assume that φ has minimal depth among the lifts of ϕ.

Theorem 2.1. *Let* φ *be an irreducible totally ramified representation* $\mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$*, let* $\phi : \mathbf{W}_K \to \mathrm{PGL}_2(\mathbb{C})$ *be its projection. Then we have*

$$d(\varphi) > d(\phi).$$

Proof. *Primitive representations.* Let φ be primitive. Put $E = K_{\mathrm{s}}^{\ker \phi}$ and $E^+ = K_{\mathrm{s}}^{\ker \varphi}$. By [3, §42.3] there exists a unique intermediate field $K \subset L \subset E$ such that E/L is a wildly ramified biquadratic extension. Then $\phi(\mathrm{Gal}(E/L))$ is a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ isomorphic to the Klein four group. Up to conjugacy $\mathrm{PGL}_2(\mathbb{C})$ has only one such subgroup. After a suitable change of basis, we may assume that it is

$$D_2 := \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \right\} \subset \mathrm{PGL}_2(\mathbb{C}). \tag{2.8}$$

The three subextensions of E/L are conjugate under $\mathrm{Gal}(E/K)$ because the conjugation action of A_4 on its normal subgroup V_4 of order four is transitive on the nontrivial elements of V_4. Hence there is a unique $r \in \mathbb{Z}$ such that $\mathrm{Gal}(E/L)_r = \mathrm{Gal}(E/L)$ and $\mathrm{Gal}(E/L)_{r+1} = \{1\}$. In section 4.2 we will see that r is odd. We call this r the *ramification depth* of E/L.

The nontrivial elements of $\mathrm{Gal}(E/L)$ are the deepest elements of $\mathrm{Gal}(E/K)$ outside the kernel of ϕ, and therefore the depth of ϕ can be expressed in terms of r.

Let us compare this to what happens for the lift φ of ϕ. Since $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is a surjection with kernel of order 2, the pre-image of $\phi(\mathbf{W}_K)$ in $\mathrm{SL}_2(\mathbb{C})$ has order $2|\phi(\mathbf{W}_K)|$. The matrices in (2.8) do not yet form a group in $\mathrm{GL}_2(\mathbb{C})$, for that we really need the nontrivial element of $\ker(\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}))$. In other words, $\mathrm{SL}_2(\mathbb{C})$ contains a unique subgroup of order $2[E : K]$ which projects onto $\phi(\mathbf{W}_K)$. As φ has minimal depth among the lifts of ϕ, $\varphi(\mathbf{W}_K)$ is precisely this subgroup. Thus $[E^+ : E] = 2$ and $\mathrm{Gal}(E^+/K)$ is a nontrivial index two central extension of $\mathrm{Gal}(E/K)$. In particular $\mathrm{Gal}(E^+/L)$ is isomorphic to the quaternion group of order eight.

Choose a subset $\{w_1 = 1, w_2, w_3, w_4\} \subset \mathrm{Gal}(E^+/L)$ which projects onto $\mathrm{Gal}(E/L)$. We may assume that the $\varphi(w_i)$ are ordered as in (2.8). As $\ker(\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}))$ is central,

$$[\varphi(w_3), \varphi(w_4)] = \left[\left(\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C}).$$

Write

$$z = [w_3, w_4] \in \mathrm{Gal}(E^+/L), \tag{2.9}$$

so that $\varphi(z) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. It follows from the definition of r and the condition on φ that

$$\mathrm{Gal}(E^+/L)_r = \mathrm{Gal}(E^+/L) \text{ and } \mathrm{Gal}(E^+/L)_{r+1} = \mathrm{Gal}(E^+/E).$$

By [14, Proposition IV.2.10] $z \in \mathrm{Gal}(E^+/L)_{2r+1}$. Now $z \notin \ker(\varphi)$ and it lies deeper in $\mathrm{Gal}(E^+/K)$ than w_2, w_3 and w_4. On the other hand, z does lie in the kernel of ϕ, which explains why φ has larger depth than ϕ.

In the sequel of this section, we assume that the depth of the element z defined in (2.9) is exactly $2r+1$. This is allowed because, in the above setting, it constitutes the worst possible case for the theorem.

Octahedral representations. Let φ be octahedral, that is, it is primitive and $\phi(\mathbf{W}_K) \cong S_4$. Let Ad denote the adjoint representation of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}(\mathrm{PGL}_2(\mathbb{C}))$. Then $\mathrm{Ad}\circ\phi$ is an irreducible 3-dimensional representation of $\mathbf{W}_K$. Since $\mathrm{PGL}_2(\mathbb{C})$ is the adjoint group of $\mathfrak{sl}_2(\mathbb{C})$, $\mathrm{Ad}\circ\phi$ has the same kernel and hence the same depth as ϕ.

By [3, Theorem 42.2] L/K is Galois with automorphism group S_3 and residue degree 2. Thus $\mathrm{Ad}(\phi(\mathbf{I}_K)) \subset \mathrm{Ad}(\phi(\mathbf{W}_K))$ is a normal subgroup of index two, isomorphic to A_4. As L/K has tame ramification index 3, the image of the wild inertia subgroup $\mathbf{P}_K$ under $\mathrm{Ad}\circ\phi$ equals the image of $\mathrm{Gal}(E/L)$. By our convention (2.8) it is $\mathrm{Ad}(D_2)$. By the definition of r as the ramification depth of E/L, we have

$$g_0 = 12,\ g_1 = \cdots = g_r = 4 \text{ and } g_{r+1} = 1$$

With the formula (2.6) we find

$$a(\mathrm{Ad}\circ\phi) = \frac{3}{12}(12 + r\cdot 4) = 3 + r,$$

and from (2.7) we conclude that

$$d(\phi) = d(\mathrm{Ad}\circ\phi) = r/3.$$

On the other hand, φ is an irreducible two-dimensional representation of $\mathbf{W}_K$, and we must base our calculations on the Galois group of E^+/K. The numbers

$$g_j = |\mathrm{Gal}(E^+/K)_j| = |\varphi(\mathrm{Gal}(E^+/K)_j)|$$

can be computed from those for ϕ by means of the twofold covering $\varphi(\mathbf{W}_K) \to \phi(\mathbf{W}_K)$. We find

$$g_0 = 24, g_1 = \cdots = g_r = 8 \text{ and } g_{r+1} = \cdots = g_{2r+1} = 2.$$

Assuming that the depth of z is precisely $2r+1$ (see above), we can also say that $g_{2r+2} = 1$. Then (2.6) gives

$$a(\varphi) = \frac{2}{24}(24 + r\cdot 8 + (r+1)\cdot 2) = 2 + \frac{5r+1}{6}.$$

Now (2.7) says that

$$d(\varphi) = (5r+1)/12.$$

We note that this is strictly larger than $d(\phi) = r/3$. As $a(\phi) \in \mathbb{Z}_{\geq 0}$, we must have $r-1 \in 6\mathbb{Z}$. This means that above not all biquadratic extensions can occur.

Tetrahedral representations. Let φ be tetrahedral, that is, it is primitive and $\phi(\mathbf{W}_K) \cong A_4$. By [3, Theorem 42.2] L/K is a cubic Galois extension. It is of prime order, so either it is unramified or it is totally ramified.

First we consider the case that L/K ramifies totally. Then $\mathbf{I}_K$ surjects onto $\mathrm{Gal}(E/K)$, so $\varphi(\mathbf{I}_K) = \varphi(\mathbf{W}_K)$. This means that within $\mathbf{I}_K$ everything is similar to octahedral representations. The same calculations as above show that

$$d(\phi) = r/3 < d(\varphi) = (5r+1)/12.$$

Now we look at the case where L/K is unramified. Then

$$\phi(\mathbf{I}_K) = \phi(\mathrm{Gal}(E/K)) = D_2.$$

To compute the depth, we replace ϕ by the 3-dimensional representation $\mathrm{Ad}\circ\phi$ of $\mathbf{W}_K$ on $\mathfrak{sl}_2(\mathbb{C})$. With r as before, $g_0 = \cdots = g_r = 4$ and $g_{r+1} = 1$. With (2.6) and (2.7) we calculate

$$a(\mathrm{Ad}\circ\phi) = \frac{3}{4}((r+1)\cdot 4) = 3(r+1),$$
$$d(\phi) = d(\mathrm{Ad}\circ\phi) = \frac{3(r+1)}{3} - 1 = r.$$

Like in the octahedral case, the numbers $\mathrm{Gal}(E^+/K)_j$ for φ are related to those for ϕ via the twofold covering $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. We find

$$g_0 = \cdots = g_r = 8 \text{ and } g_{r+1} = \cdots = g_{2r+1} = 2.$$

Moreover $g_{2r+2} = 1$ if we assume that the depth of z is $2r+1$. Now (2.6) says

$$a(\varphi) = \frac{2}{8}\big((r+1)\cdot 8 + (r+1)\cdot 2\big) = 5(r+1)/2 \in \mathbb{Z},$$

and from (2.7) we obtain

$$d(\varphi) = \frac{5(r+1)}{2\cdot 2} - 1 = \frac{5r+1}{4}.$$

Again, this is larger than $d(\phi) = r$.

Imprimitive representations. Consider an imprimitive totally ramified representation $\varphi : \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$. By [3, §41.4] there exists a separable totally ramified quadratic extension L/K and a character ξ of $\mathbf{W}_L$ such that $\varphi = \mathrm{ind}_{\mathbf{W}_L}^{\mathbf{W}_K}(\xi)$. Let $\mathfrak{p}_K^{1+d(L/K)}$ be the discriminant of L/K. If $L \cong K[X]/(X^2+X+b)$, then one deduces from [3, §41.1] that $d(L/K) = -\nu_K(b) > 0$.

From the proof of [3, Lemma 41.5] one sees that the level of φ equals $d(\xi) + d(E/F)$. By construction the level of a n-dimensional irreducible representation of $\mathbf{W}_K$ equals n times its depth, so

$$d(\varphi) = \big(d(\xi) + d(L/K)\big)/2. \tag{2.10}$$

As before we assume that φ is minimal among the lifts of ϕ. Then [3, §41.4] says that $d(\xi) > d(L/K)$, and in particular $d(\xi) \geq 2$. Since $\mathrm{Gal}(K_s/L)^2$ is a pro-2-group, the image of ξ in $\mathbb{C}^\times$ is a subgroup of even order.

Let σ be the nontrivial element of $\mathrm{Gal}(L/K)$, so that the restriction of φ to $\mathbf{W}_L$ is $\xi \oplus \sigma(\xi)$. If $\xi(w) = -1$, then also $\xi(\sigma(w)) = -1$. As $\xi(\mathbf{W}_L)$ is even, this means that $\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \phi(\mathbf{W}_L)$. We note that, as every $\mathbf{W}_K \setminus \mathbf{W}_L$ interchanges ξ and $\sigma(\xi)$, the kernel of ϕ equals the kernel of $\xi \oplus \sigma(\xi)$ composed with the projection $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. Thus the kernel of ϕ contains the kernel of φ with index two. More precisely

$$\ker(\phi) = (\xi \oplus \sigma(\xi))^{-1}\big\{ \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \big\} = \xi^{-1}\{1, -1\} = \ker(\xi^2).$$

By the same argument as above also $\ker(\mathrm{ind}_{\mathbf{W}_L}^{\mathbf{W}_K}\xi^2) = \ker(\xi^2)$. Hence ϕ and $\mathrm{ind}_{\mathbf{W}_L}^{\mathbf{W}_K}(\xi^2)$ have the same kernel, and in particular the same depth. With (2.10) we can express it as

$$d(\phi) = \big(d(\xi^2) + d(L/K)\big)/2. \tag{2.11}$$

The depth (or level) of ξ is the least l such that ξ (or rather its composition with the Artin reciprocity isomorphism) is nontrivial on the higher units group $U_L^l = 1 + \mathfrak{p}_L^l \subset L^\times$. For $l > 0$ the group U_L^l/U_L^{l+1} has exponent 2, so $\xi(U_L^{d(\xi)}) = \{1, -1\}$. Consequently $U_L^{d(\xi)} \subset \ker \xi^2$ and $d(\xi^2) < d(\xi)$. Comparing (2.10) and (2.11), we get

$$d(\varphi) - d(\phi) = \big(d(\xi) - d(\xi^2)\big)/2 > 0.$$

3. *L*-packets

According to a classical result of Shelstad [15, p.200], for F of characteristic zero all the L-packets $\Pi_\varphi(\mathrm{SL}_2(F))$ have cardinality $1, 2$ or 4. We will check below, after (3.3), that the same holds for the L-packets for $\mathrm{SL}_2(K)$. It will follow from the classification in this section that L-packets for $\mathrm{SL}_1(D)$ have cardinality 1 or 2.

Theorem 3.1. [1] *Let $\phi : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ be an L-parameter for $\mathrm{SL}_2(K)$, and let $\varphi : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ be a lift of minimal depth. For any π in one of the L-packets $\Pi_\varphi(\mathrm{GL}_2(K))$, $\Pi_\varphi(GL_1(D))$, $\Pi_\phi(\mathrm{SL}_2(K))$ and $\Pi_\phi(\mathrm{SL}_1(D))$:*

$$d(\phi) \leq d(\varphi) = d(\pi).$$

Moreover $d(\phi) = d(\varphi) = d(\pi)$ if φ is essentially tame, in particular whenever φ is unramified.

We define the groups

$$\begin{aligned}
&C(\phi) := Z_{\mathrm{SL}_2(\mathbb{C})}(\mathrm{im}\ \phi),\\
&\mathcal{S}_\phi := C(\phi)/C(\phi)^\circ = \pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi)),\\
&\mathcal{Z}_\phi := \mathrm{Z}(\mathrm{SL}_2(\mathbb{C}))/\mathrm{Z}(\mathrm{SL}_2(\mathbb{C})) \cap C(\phi)^\circ,\\
&S_\phi := \pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)).
\end{aligned} \tag{3.1}$$

The group S_ϕ is abelian, $\mathcal{S}_\phi$ can be nonabelian, and there is a short exact sequence

$$1 \to \mathcal{Z}_\phi \to \pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi)) \to \pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)) \to 1. \tag{3.2}$$

It is easily seen that $|\mathcal{Z}_\phi| = 2$ if and only if ϕ is relevant for $\mathrm{SL}_1(D)$. By [2, Theorem 3.3] there are bijections

$$\begin{aligned}
&\mathbf{Irr}\big(\pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi))\big) \longleftrightarrow \Pi_\phi(\mathrm{SL}_2(K)),\\
&\mathbf{Irr}\big(\pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi))\big) \longleftrightarrow \Pi_\phi(\mathrm{SL}_2(K)) \cup \Pi_\phi(\mathrm{SL}_1(D)).
\end{aligned} \tag{3.3}$$

We remark that for $\mathrm{SL}_2(F)$ with $\mathrm{char}(F) = 0$, (3.3) was shown in [9, Theorem 4.2] and [11, Theorem 12.7]. Recall that $\mathfrak{T}(\varphi)$ is the abelian group of characters χ of $\mathbf{W}_K$ with $\varphi \otimes \chi \cong \varphi$. By [9, Theorem 4.3] and by [2, (21)]

$$\mathfrak{T}(\varphi) \cong \pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)). \tag{3.4}$$

By [3, Proposition 41.3], and by the classification of L-parameters for the principal series in Subsection 3.2, $\mathfrak{T}(\varphi)$ has order dividing four. This shows that all L-packets for $\mathrm{SL}_2(K)$ have order 1, 2 or 4.

3.1. *Stability*

Before we proceed with the classification of L-packets, some remarks about the stability of the associated distributions are in order. In this subsection K can be any local non-archimedean field. Recall that a class function on an algebraic K-group $\mathcal{G}(K)$ is called stable if it is constant on the intersection of any $\mathcal{G}(K_s)$-conjugacy class with $\mathcal{G}(K)$. For an invariant distribution on $\mathcal{G}(K)$ one would like to use a similar definition of stability, but that does not work well in general. Instead, stable distributions are usually defined in terms of stable orbital integrals. But, whenever an invariant distribution δ on $\mathcal{G}(K)$ is represented by a class function on an open dense subset of $\mathcal{G}(K)$, we can use the easier criterion for stability of functions to determine whether or not δ is stable.

Harish-Chandra proved that the trace of an admissible representation is a distribution which is represented by a locally constant function on the set of regular semisimple elements of $\mathcal{G}(K)$, see [7]. So the study the stability of traces of $\mathcal{G}(K)$-representations, it suffices to look at (regular) semisimple elements of $\mathcal{G}(K)$.

For semisimple elements in $\mathrm{GL}_2(K)$ conjugacy is the same as stable conjugacy, it is determined by characteristic polynomials. Hence every irreducible (admissible) representation of $\mathrm{GL}_2(K)$ defines a stable distribution.

The semisimple conjugacy classes in $\mathrm{GL}_1(D)$ are naturally in bijection with the elliptic conjugacy classes in $\mathrm{GL}_2(K)$, i.e. those semisimple classes for which the characeristic polynomials are irreducible over K. Moreover any irreducible essentially square-integrable representation of $\mathrm{GL}_2(K)$ is already determined by the values of its trace on elliptic elements. These observations constitute some of the foundations of the Jacquet–Langlands correspondence [12]. In fact the Jacquet–Langlands correspondence can be defined as the unique bijection between $\mathbf{Irr}(\mathrm{GL}_1(D))$ and the essentially square-integrable representations in $\mathbf{Irr}(\mathrm{GL}_2(K))$ which preserves the traces on elliptic conjugacy classes, up to a sign. Consequently the trace of any irreducible representation π of $\mathrm{GL}_1(D)$ is the restriction of a stable distribution on $\mathrm{GL}_2(K)$ to the set of elliptic elements. In particular the trace of π is itself a stable distribution.

Theorem 3.2. *Let* ϕ *be a L-parameter for* $\mathrm{SL}_2(K)$.

- *(a) Write* $\Pi_\phi(\mathrm{SL}_2(K)) = \{\pi_1, \dots, \pi_m\}$. *The trace of* $\pi := \pi_1 \oplus \cdots \oplus \pi_m$ *is a stable distribution on* $\mathrm{SL}_2(K)$. *Any other stable distribution that can be obtained from* $\Pi_\phi(\mathrm{SL}_2(K))$ *is a scalar multiple of the trace of* π.
- *(b) Suppose that* ϕ *is relevant for* $\mathrm{SL}_1(D)$ *and write* $\Pi_\phi(\mathrm{SL}_1(D)) = \{\pi'_1, \dots, \pi'_{m'}\}$. *The trace of* $\pi' := \pi'_1 \oplus \cdots \oplus \pi'_{m'}$ *is a stable distribution on* $\mathrm{SL}_1(D)$. *Any other stable distribution that can be obtained from* $\Pi_\phi(\mathrm{SL}_1(D))$ *is a scalar multiple of the trace of* π'.

Proof. (a) Since the restriction of irreducible representations from $\mathrm{GL}_2(K)$ to $\mathrm{SL}_2(K)$ is multiplicity-free [5, §1], $\pi = \pi_1 \oplus \cdots \oplus \pi_m$ is the restriction of some irreducible representation of $\mathrm{GL}_2(K)$. If $\varphi : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is any lift of ϕ, the image of ϕ under the local Langlands correspondence is such a representation. We denote this representation of $\mathrm{GL}_2(K)$ again by π. By the above remarks, its trace is a stable distribution on $\mathrm{GL}_2(K)$, and hence also on $\mathrm{SL}_2(K)$.

The different π_i are inequivalent, but they are $\mathrm{GL}_2(K)$ conjugate, because π is irreducible. If a linear combination $\sum_{i=1}^m \lambda_i \mathrm{tr}(\pi_i)$ is a stable distribution, then it must be invariant under conjugation by $\mathrm{GL}_2(K)$. Hence all the $\lambda_i \in \mathbb{C}$ must be equal.

(b) The restriction of representations from $\mathrm{GL}_1(D)$ to $\mathrm{SL}_1(D)$ can have multiplicities, but still every constituent will appear with the same multiplicity [9, Lemma 2.1.d]. So there exists an integer μ such that $\mu\pi' = \mu\pi'_1 \oplus \cdots \oplus \mu\pi'_{m'}$ lifts to an irreducible representation of $\mathrm{GL}_1(D)$. The L-parameter of such a representation is a lift of ϕ, so we can take $\mathrm{JL}(\pi)$, the image of π under the Jacquet–Langlands correspondence.

As remarked above, $\mathrm{tr}(\mathrm{JL}(\pi))$ is stable distribution on $\mathrm{GL}_1(D)$ and by restriction also on $\mathrm{SL}_1(D)$. Thus $\mathrm{tr}(\pi') = \mu^{-1}\mathrm{tr}(\mathrm{JL})(\pi)$ is also a stable distribution on $\mathrm{SL}_1(D)$. By the same argument as for part (a), any linear combination of the tr (π'_i) which is stable, must be a scalar multiple of $\mathrm{tr}(\pi')$.

We remark that Theorem 3.2 also holds for inner forms of $\mathrm{SL}_n(F)$ with $n > 2$. The proof is the same, one only has to replace the elliptic conjugacy classes by the conjugacy classes that correspond to elements of that particular inner form.

3.2. *L-packets of cardinality one*

First we consider the case that $\varphi : \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$ is irreducible, so the L-packet consists of supercuspidal representations. By (3.4) and (3.3), $\Pi_\phi(\mathrm{SL}_2(K))$ is a singleton if and only if φ is primitive. The L-parameter ϕ is relevant for $\mathrm{SL}_1(D)$, so $\Pi_\phi(\mathrm{SL}_1(D))$ is nonempty. It follows from (3.3) and (3.2) that $\mathcal{Z}_\phi \cong \pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi)) \cong \mathbb{Z}/2\mathbb{Z}$, and then from (3.3) that $\Pi_\phi(\mathrm{SL}_1(D))$ is also a singleton. Any primitive representation of $\mathbf{W}_K$ is either octahedral or tetrahedral, as in Section 2. See [3, §42] for more background.

Suppose now that $\varphi : \mathbf{W}_K \to \mathrm{GL}_2(\mathbb{C})$ is reducible, so ϕ is a L-parameter for the principal series of $\mathrm{SL}_2(K)$. If $\phi(\mathbf{W}_K) = 1$ and $\phi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is the canonical projection, then ϕ is relevant for $\mathrm{SL}_1(D)$. In this case $\Pi_\phi(\mathrm{SL}_1(D))$ is just the trivial representation of $\mathrm{SL}_1(D)$, and $\Pi_\phi(\mathrm{SL}_2(K))$ consists of the Steinberg representation of $\mathrm{SL}_2(K)$ – the unique irreducible square-integrable, non-supercuspidal representation.

All other principal series L-parameters are trivial on $\mathrm{SL}_2(\mathbb{C}))$ and are irrelevant for $\mathrm{SL}_1(D)$. By conjugating ϕ, we may assume that its image is contained in the diagonal torus of $\mathrm{PGL}_2(\mathbb{C})$. One checks that $Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)$ is connected unless the image of ϕ is $\{1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. Whenever $Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)$ is disconnected, its L-packet has two elements, see Subsection 3.5.

If $Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)$ is connected, then $\Pi_\phi(\mathrm{SL}_2(K))$ consists of precisely one principal series representation. Let T be the diagonal torus of $\mathrm{SL}_2(K)$, and let χ_ϕ be the character of T determined by local class field theory. Then $\Pi_\phi(\mathrm{SL}_2(K))$ is the Langlands quotient of the parabolic induction of χ_ϕ, and the depth of that representation equals the depth of χ_ϕ.

3.3. *Supercuspidal L-packets of cardinality two*

For such L-parameters (3.4) shows that

$$\mathfrak{T}(\varphi) \cong \pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\phi)) \cong \mathbb{Z}/2\mathbb{Z}.$$

The L-parameter ϕ is relevant for $\mathrm{SL}_1(D)$, so by (3.2) $|\pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi))| = 4$. Then $\pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi))$ is either $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. In any case, it is abelian and has precisely four inequivalent characters. Now (3.3) says that

$$|\Pi_\phi(\mathrm{SL}_1(D))| = |\Pi_\phi(\mathrm{SL}_2(K))| = 2.$$

Now we classify the discrete L-parameters ϕ for which the packet $\Pi_\phi(\mathrm{SL}_2(K))$ is not a singleton. We note that every L-parameter for a supercuspidal representation of $\mathrm{SL}_2(K)$ has to be trivial on $\mathrm{SL}_2(\mathbb{C})$. For if it were nontrivial on $\mathrm{SL}_2(\mathbb{C})$, then the image of $\mathbf{W}_K$ would be in the centre of $\mathrm{PGL}_2(\mathbb{C})$, and we would get the L-parameter for the Steinberg representation, as discussed in the previous subsection. Since we want ϕ to be discrete, it has to be an irreducible projective two-dimensional representation of $\mathbf{W}_K$.

Let φ be an irreducible two-dimensional representation of $\mathbf{W}_K$ which lifts ϕ. By (3.4) and (3.3) the associated L-packet $\Pi_\phi(\mathrm{SL}_2(K))$ has more than one element if and only if φ is imprimitive. By [3, §41.3] φ is imprimitive if and only if there exists a separable quadratic extension E/K and a character ξ of $E^\times$ such that $\varphi \cong \mathrm{Ind}_{E/K}\xi$. By the irreducibility $\xi^\sigma \neq \xi$, where σ is the nontrivial automorphism of E over K.

Lemma 3.3. *Let ϕ and $\varphi \cong \mathrm{Ind}_{E/K}\xi$ be as above.*

(a) *Suppose that the character* $\xi^\sigma\xi^{-1}$ *of* $E^\times$ *has order two. Then* φ *is triply imprimitive and there exists a biquadratic extension* L/K *such that* $\ker(\phi) = \mathbf{W}_L$ *and* $L \supset E$.

(b) *Suppose that* $\xi^\sigma\xi^{-1}$ *has order* > 2. *Then* φ *is simply imprimitive.*

Proof. Let χ_E be the unique character of $\mathbf{W}_K$ with kernel $\mathbf{W}_E$. Then $\chi_E \in \mathfrak{T}(\varphi)$, this holds in general for induction of irreducible representations from subgroups of index two. In particular $|\mathfrak{T}(\varphi)| \in \{2, 4\}$. From [3, Corollary 41.3] we see that $\mathfrak{T}(\varphi) = \{1, \chi_E\}$ if and only if the character $\xi^\sigma\xi^{-1}$ of $\mathbf{W}_E$ cannot be lifted to a character of $\mathbf{W}_F$. Since the target group $\mathbb{C}^\times$ is divisible, this happens if and only if $\xi^\sigma\xi^{-1}$ does not equal

$$(\xi^\sigma\xi^{-1})^\sigma = \xi\xi^{-\sigma} = (\xi^\sigma\xi^{-1})^{-1}.$$

We conclude that the representation $\varphi = \mathrm{Ind}_{E/K}\xi$ is triply imprimitive if $\xi^\sigma\xi^{-1}$ has order two and is simply imprimitive otherwise.

Now we focus on the triply imprimitive case. By local class field theory there exists a unique separable quadratic extension L/E such that $\xi^\sigma\xi^{-1}$ is the associated character χ_L of $E^\times$. We consider it also as a character of $\mathbf{W}_E$. Then

$$\mathbf{W}_L = \ker(\chi_L) = \{w \in \mathbf{W}_K : \varphi(w) \in Z(\mathrm{GL}_2(\mathbb{C}))\}.$$

Hence $\mathbf{W}_L = \ker(\phi)$ is a normal subgroup of $\mathbf{W}_K$, which means that L/K is a Galois extension. The explicit form of φ entails that the image of ϕ is the Klein four group. Consequently

$$\mathrm{Gal}(L/K) \cong \mathbf{W}_K/\mathbf{W}_L \cong \phi(\mathbf{W}_K) \cong (\mathbb{Z}/2\mathbb{Z})^2, \tag{3.5}$$

which says that L/K is biquadratic.

We remark that the depth of $\varphi = \mathrm{Ind}_{E/K}\xi$ can be computed in the same way as for the imprimitive representations in Section 2, see in particular (2.10).

3.4. *Supercuspidal L-packets of cardinality four*

We continue with the case when φ is triply imprimitive, as in (3.5). This means that we have a biquadratic extension L/K and the Langlands parameter

$$\phi : W_K \to \mathrm{Gal}(L/K) \cong (\mathbb{Z}/2\mathbb{Z})^2 \subset \mathrm{PGL}_2(\mathbb{C}). \tag{3.6}$$

We also have

$$Z_{\mathrm{PGL}_2(\mathbb{C})}(\mathrm{im}\,\phi) = \pi_0(Z_{\mathrm{PGL}_2(\mathbb{C})}(\mathrm{im}\,\phi)) = S_\phi \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

This implies, by (3.3), that $\Pi_\phi(\mathrm{SL}_2(K))$ is a supercuspidal packet of cardinality 4.

We note the isomorphism $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$, and the morphism

$$\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C}).$$

As in [16, §14], the pull-back $\mathcal{S}_\phi$ of S_ϕ is isomorphic to the group of unit quaternions $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. This group admits four characters and one irreducible representation of degree 2. Only the two-dimensional representation ρ_0 has nontrivial central character.

The parameter ϕ creates a packet with five elements, which are allocated to $\mathrm{SL}_2(K)$ or $\mathrm{SL}_1(D)$ according to central characters. So ϕ gives rise to an L-packet $\Pi_\phi(\mathrm{SL}_2(K))$ with 4 elements, and a singleton packet to the inner form $\mathrm{SL}_1(D)$.

Theorem 3.4. *Let* L/K *be a biquadratic extension, let* ϕ *be the Langlands parameter (3.6). If* t *is the highest break in the upper ramification of* $\mathrm{Gal}(L/K)$ *then* $d(\phi) = t$. *The allowed values of* $d(\phi)$ *are* $1, 3, 5, 7, \ldots$ *except in Case 2.2 (see 4.2 in Section 4), when the allowed values are* $3, 5, 7, \ldots$.

Proof. From the inclusion $L \subset K_s$ we obtain a natural surjection

$$\pi_{L/K}\colon \mathrm{Gal}(K_s/K) \to \mathrm{Gal}(L/K).$$

Let K_{ur} be the maximal unramified extension of K in K_s and let K_{ab} be the maximal abelian extension of K in K_s. We have a commutative diagram, where the horizontal maps are the canonical maps and the vertical maps are the natural projections

$$\begin{tikzcd}
1 \arrow[r] & I_{K_s/K} \arrow[r, "\iota_1"] \arrow[d, "\alpha_1"'] & \mathrm{Gal}(K_s/K) \arrow[r, "p_1"] \arrow[d, "\pi_1"'] & \mathrm{Gal}(K_{\mathrm{ur}}/K) \arrow[r] \arrow[d, "id"'] & 1 \\
1 \arrow[r] & I_{K_{\mathrm{ab}}/K} \arrow[r, "\iota_2"] \arrow[d, "\alpha_2"'] & \mathrm{Gal}(K_{\mathrm{ab}}/K) \arrow[r, "p_2"] \arrow[d, "\pi_2"'] & \mathrm{Gal}(K_{\mathrm{ur}}/K) \arrow[r] \arrow[d, "\beta"'] & 1 \\
1 \arrow[r] & \mathbf{I}_{L/K} \arrow[r, "\iota_3"] & \mathrm{Gal}(L/K) \arrow[r, "p_3"] & \mathrm{Gal}(L \cap K_{\mathrm{ur}}/K) \arrow[r] & 1
\end{tikzcd}$$

In the above notation, we have $\pi_{L/K} = \pi_2 \circ \pi_1$. Let

$$\cdots \subset \mathbf{I}^{(2)} \subset \mathbf{I}^{(1)} \subset \mathbf{I}^{(0)} \subset G = \mathrm{Gal}(L/K) \tag{3.7}$$

be the filtration of the relative inertia subgroup $\mathbf{I}^{(0)} = \mathbf{I}_{L/K}$ of $\mathrm{Gal}(L/K)$, $\mathbf{I}^{(1)}$ is the wild inertia subgroup, and so on. Note that $\mathbf{I}^{(r)}$ is the restriction of the filtration G^r of $G = \mathrm{Gal}(L/K)$ to the subgroup $\mathbf{I}_{L/K}$, i.e, $\mathbf{I}^{(r)} = \iota_3(G^r)$. Let

$$\cdots \subset I^{(2)} \subset I^{(1)} \subset I^{(0)} \subset G = \mathrm{Gal}(\overline{K}/K) \tag{3.8}$$

be the filtration of the absolute inertia subgroup $I^{(0)} = I_{K_s/K}$ of $\mathrm{Gal}(K^s/K)$, $I^{(1)}$ is the wild inertia subgroup, and so on.

We have

$$(\forall r)\ \pi_{L/K} I^{(r)} = \mathbf{I}^{(r)} \tag{3.9}$$

This follows immediately from the above diagram. Here, we identify $I^{(r)}$ with $\iota_1(I^{(r)})$ and $\mathbf{I}^{(r)}$ with $\iota_3(\mathbf{I}^{(r)})$. (Note that α is *injective*. Therefore, by (3.9), we have

$$\phi(I^{(r)}) = 1 \iff (\alpha \circ \pi_{L/K})(I^{(r)}) = 1 \iff \alpha(\mathbf{I}^{(r)}) = 1 \iff \mathbf{I}^{(r)} = 1.$$

The *highest break* t has the property that $I^{(t+1)} = 1$ and $I^{(t)} \neq I^{(t+1)}$. It follows that $d(\phi) = t$.

Case 1**:** There are two ramification breaks occurring at -1 and some odd integer $t > 0$:

$$\{1\} = \cdots = \mathbf{I}^{(t+1)} \subset \mathbf{I}^{(t)} = \cdots = \mathbf{I}^{(0)} = \mathbf{I}_{L/K} \subset \mathrm{Gal}(L/K), \quad d(\phi) = t.$$

The allowed depths are 1, 3, 5, 7, ...

Case 2.1**:** One single ramification break occurs at some odd integer $t > 0$:

$$\{1\} = \cdots = \mathbf{I}^{(t+1)} \subset \mathbf{I}^{(t)} = \cdots = \mathbf{I}^{(0)} = \mathbf{I}_{L/K} = \mathrm{Gal}(L/K); \quad d(\phi) = t.$$

The allowed depths are $1, 3, 5, 7, \ldots$.

Case 2.2**:** There are two ramification breaks occurring at some odd integers $t_1 < t_2$ (with $\mathbf{I}^{(0)} = \mathbf{I}_{L/K}$) :

$$\{1\} = \cdots = \mathbf{I}^{(t_2+1)} \subset \mathbf{I}^{(t_2)} = \cdots = \mathbf{I}^{(t_1+1)} \subset \mathbf{I}^{(t_1)} = \cdots = \mathbf{I}^{(0)} = \mathbf{I}_{L/K} = \mathrm{Gal}(L/K);$$

$$d(\phi) = t_2.$$

The allowed depths are $3, 5, 7, 9, \ldots$.

Theorem 3.4 contrasts with the case of $\mathrm{SL}_2(\mathbb{Q}_p)$ with $p > 2$. Here there is a unique biquadratic extension L/K, and the associated L-parameter $\phi : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$ has depth zero.

3.5. *Principal series L-packets of cardinality two*

Recall from Subsection 3.2 that a principal series L-parameter whose L-packet is not a singleton has image $\{1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ in the diagonal torus $T^\vee$ of $\mathrm{PGL}_2(\mathbb{C})$. Thus it comes from a character $\mathbf{W}_K \to \mathbb{C}^\times$ of order two. Define

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to K^\times$$

to be the projection $(g, M) \mapsto g$ followed by the Artin reciprocity map

$$\mathbf{a}_K : \mathbf{W}_K \to K^\times.$$

Let E/K be a quadratic extension and let χ_E be the associated quadratic character of $K^\times$. Consider the map

$$K^\times \to \mathrm{PGL}_2(\mathbb{C}), \qquad \alpha \mapsto \begin{pmatrix} \chi_E(\alpha) & 0 \\ 0 & 1 \end{pmatrix}$$

The composite map

$$\phi_E : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to K^\times \to \mathrm{PGL}_2(\mathbb{C})$$

is then an L-parameter attached to χ_E. For the centralizer of the image, we have

$$Z_{\mathrm{PGL}_2(\mathbb{C})}(\mathrm{im}\,\phi_E) = N_{\mathrm{PGL}_2(\mathbb{C})}(T^\vee), \quad S_\phi \cong \mathcal{S}_\phi = \{1, w\},$$

where w generates the Weyl group of the dual group $\mathrm{PGL}_2(\mathbb{C})$. As there are two characters $1, \epsilon$ of $W = \{1, w\}$, (3.3) says that the L-packet has cardinality two. There are two enhanced parameters $(\phi_E, 1)$ and (ϕ_E, ϵ), which parametrize the two elements in the L-packet $\Pi_{\phi_E} = \Pi_{\phi_E}(\mathrm{SL}_2(K))$. We will write

$$\Pi_{\phi_E} = \{\pi_E^1, \pi_E^2\}. \tag{3.10}$$

If $\gamma \in K_s$ is a root of $X^2 - X - \beta \in K[X]$, the quadratic extension $K(\gamma)$ is denoted also by $K(\wp^{-1}(\beta))$, with $\beta \in K$, where $\wp(X) = X^2 - X$. So the quadratic character

$$\chi_{n,j} = (-, u_j \varpi^{-2n-1} + \wp(K)]$$

is associated with the quadratic extension $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$, see (4.2) in Section 4.

Let E/K be a quadratic extension. There are two kinds: the unramified one $E_0 = K(\gamma_0)$ and countably many totally (and wildly) ramified $E = K(\gamma)$. The unramified quadratic extension has a single ramification break for $t = -1$.

Let E/K be a quadratic totally ramified extension. According to [6, Proposition 11, p.411 and Proposition 14, p.413], there is a single ramification break for $t = 2n + 1$. Each value $2n + 1$ occurs as a break, with $n \geq 0, 1, 2, 3, \ldots$. By Theorem 3.4, adapted to the present case, we have

$$d(\phi_E) = 2n + 1.$$

Fix a basis $\mathcal{B} = \{u_1, \ldots, u_f\}$ of $\mathbb{F}_q/\mathbb{F}_2$ and let $u_j \in \mathcal{B}$. The next result shows how to realise the extension E/K.

Theorem 3.5. *If $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$ then*

$$d(\phi_E) = 2n+1$$

with $n = 0, 1, 2, 3, 4, \ldots$.

Proof. Let $\mathbf{a}_K : \mathbf{W}_K \to K^\times$ be the Artin reciprocity map. Then we have [1, Theorem 3.6]:

$$\mathbf{a}_K(\mathrm{Gal}(K_\mathrm{s}/K)^l) = U^{\lceil l\rceil}$$

for all $l \geq 0$, where $\lceil l\rceil$ denotes the least integer greater than or equal to l, and U_K^i is the ith higher unit group.

We are concerned here with the quadratic character $\chi = \chi_E$ and the associated L-parameter $\phi = \phi_E$. The level $\ell(\chi)$ of χ is the least integer $n \geq 0$ for which $\chi(U_K^{n+1}) = 1$. Call this integer N. For this integer N, we have

$$N < l \leq N+1 \implies \mathbf{a}_K(\mathrm{Gal}(K_\mathrm{s}/K)^l) = U_K^{\lceil l\rceil} = U_K^{N+1} \text{ on which } \chi \text{ is trivial}$$

$$N-1 < l \leq N \implies \mathbf{a}_K(\mathrm{Gal}(K_\mathrm{s}/K)^l) = U_K^{\lceil l\rceil} = U_K^{N} \text{ on which } \chi \text{ is nontrivial}$$

The L-parameter ϕ will factor through $K^\times$ and we have to consider its depth $d(\phi)$. Recall: the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(K_\mathrm{s}/K)^l$ for all $l > d(\phi)$. Then $d(\phi) = N$ in view of the above two implications. We infer that

$$\ell(\chi_E) = d(\phi_E). \tag{3.11}$$

If χ is the unramified quadratic character given by $\chi(x) = (-1)^{\mathrm{val}_K(x)}$ then we will have to allow $N = -1$ in which case ϕ has negative depth.

If $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$ then $\chi_E = \chi_{n,j}$ and so we have

$$\ell(\chi_E) = \ell(\chi_{n,j}). \tag{3.12}$$

We now compute the level of the quadratic character $\chi_{n,j}$ defined in (4.2). Every $\alpha \in U_K^i$ has the form $\alpha = 1 + \varepsilon\varpi^i$, with $\varepsilon \in \mathfrak{o}$, and can be expanded in the convergent product

$$\alpha = \prod\nolimits_{i\geq 1}(1+\theta_i\varpi^i)$$

for unique $\theta_i \in \mathbb{F}_q$. As we can see in the proof of Theorem 4.2,

$$d_\varpi(1+\theta_{2n+1}\varpi^{2n+1}, u_j\varpi^{-2n-1}) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(u_j\theta_{2n+1})$$

and

$$d_\varpi(1+\theta_i\varpi^i, u_j\varpi^{-2n-1}) = 0$$

if $i \nmid 2n+1$. There exists, therefore, an element $\alpha \in U_K^{2n+1}$ such that $\chi_{n,j}(\alpha) \neq 0$ and $\chi_{n,j}(U_K^{2n+2}) = 1$. We infer that

$$\ell(\chi_{n,j}) = 2n+1. \tag{3.13}$$

The theorem now follows from (3.11), (3.12) and (3.13).

We conclude that, if $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$, then

$$d(\pi_E^i) \geq 2n+1$$

with $i = 1, 2$.

It follows that the depths of the irreducible representations π_E^1, π_E^2 in the L-packet Π_{ϕ_E} can be arbitrarily large. For representations of enormous depth, such as the ones encountered in this article, the term *hadopelagic* commends itself, in contrast to the currently accepted term *epipelagic* for representations of modest depth, see `en.wikipedia.org/wiki/Epipelagic`.

4. Artin-Schreier symbol

Let K be a local field of characteristic p with finite residue field k. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$. Let $\mathfrak{o}$ be the ring of integers in K and $\mathfrak{p} \subset \mathfrak{o}$ the maximal ideal. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi\mathfrak{o} \cong \varpi\mathbb{F}_q[[\varpi]]$. The group of units is denoted by $\mathfrak{o}^\times$ and ν represents a normalized valuation on K, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$.

Following [8, IV.4 - IV.5], we have the reciprocity map

$$\Psi_K : K^\times \to \mathrm{Gal}(K_{\mathrm{ab}}/K)$$

We define the map (Artin-Schreier symbol)

$$(-,-] : K^\times \times K \to \mathbb{F}_p$$

by the formula

$$(\alpha, \beta] = \Psi_K(\alpha)(\gamma) - \gamma$$

where γ is a root of the polynomial $X^p - X - \beta$. The polynomial $X^p - X$ is denoted $\wp(X)$. According to [8, p.148] the pairing $(-,-]$ determines the nondegenerate pairing

$$K^\times/K^{\times p} \times K/\wp(K) \to \mathbb{F}_p. \tag{4.1}$$

Let us fix a coset $\beta + \wp(K) \in K/\wp(K)$. According to (4.1), this coset determines an element of $\mathrm{Hom}(K^\times/K^{\times p}, \mathbb{F}_p)$.

Now specialise to $p = 2$. We will identify the additive group $\mathbb{F}_2$ with the multiplicative group $\mu_2(\mathbb{C}) = \{1, -1\} \subset \mathbb{C}$. In that case, the elements of $\mathrm{Hom}(K^\times/K^{\times 2}, \mathbb{F}_2)$ are precisely the quadratic characters of $K^\times$. Since the pairing (4.1) is nondegenerate, the quadratic characters are parametrised by the cosets $\beta + \wp(K) \in K/\wp(K)$. Now the index of $\wp(K)$ in K is infinite; in fact, the powers $\{\varpi^{-2n-1} : n \geq 0\}$ are distinct coset representatives, see [8, p.146].

Lemma 4.1. *For* $K = \mathbb{F}_2((\varpi))$ *the set of powers* $\{\varpi^{-2n-1} : n \geq 0\}$ *is a complete set of coset representatives.*

That is not the case when $K = \mathbb{F}_q((\varpi))$ has residue degree $f > 1$. Let $\mathcal{B} = \{u_1, \ldots, u_f\}$ denote a basis of the $\mathbb{F}_2$-linear space $\mathbb{F}_q$. Then,

$$\{u_j\varpi^{-2n-1} : n \geq 0, j = 1, \ldots, f\}$$

is a complete set of coset representatives of $K/\wp(K)$, see §5 and §6 of [6].

The pairing (4.1) creates a sequence of quadratic characters

$$\chi_{n,j}(\alpha) := (\alpha, u_j\varpi^{-2n-1} + \wp(K)] \tag{4.2}$$

with $n \geq 0$ and $j = 1, \ldots, f$.

4.1. *Explicit formula for the Artin-Schreier symbol*

In [8, Corollary 5.5, p.148], the authors introduce the map d_ϖ which we now describe. Let ϖ be a fixed uniformizer. Using the isomorphism $K = \mathbb{F}_q((\varpi))$, where $q = 2^f$, every element $\alpha \in K$ can be uniquely expanded as

$$\alpha = \sum_{i \geq i_a} \vartheta_i \varpi^i \, , \;\; \vartheta_i \in \mathbb{F}_q. \tag{4.3}$$

Put

$$\frac{d\alpha}{d\varpi} = \sum_{i \geq i_a} i\vartheta_i \varpi^i \, , \;\; \mathrm{res}_\varpi(\alpha) = \vartheta_{-1}.$$

Define the pairing

$$d_\varpi : K^\times \times K \to \mathbb{F}_2 \, , \;\; d_\varpi(\alpha, \beta) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \mathrm{res}_\varpi(\beta \alpha^{-1} \frac{d\alpha}{d\varpi}) \tag{4.4}$$

By [8, Theorem 5.6. p.149], the pairing $(-,-]$ coincides with the pairing defined in (4.4). In particular, d_ϖ does not depend on the choice of uniformizer.

We conclude that every quadratic character $\chi_{n,j}$ from (4.2) is completely described by

$$d_\varpi(\alpha, u_j \varpi^{-2n-1}) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \mathrm{res}_\varpi(u_j \varpi^{-2n-1} \alpha^{-1} \frac{d\alpha}{d\varpi}) \, , \;\; n \geq 0. \tag{4.5}$$

We seek a formula more explicit than (4.5).

By [8, Proposition 5.10, p. 17], for every $\alpha \in K^\times$ there exist uniquely determined $k \in \mathbb{Z}$ and $\theta_i \in \mathbb{F}_q$ for $i \geq 0$ such that α can be expanded in the convergent product

$$\alpha = \varpi^k \theta_0 \prod_{i \geq 1} (1 + \theta_i \varpi^i) \tag{4.6}$$

We have

$$\begin{aligned} d_\varpi(\varpi^k \theta_0 \prod_{i \geq 1} (1 + \theta_i \varpi^i), u_j \varpi^{-2n-1}) = & \\ & d_\varpi(\theta_0 \varpi^k, u_j \varpi^{-2n-1}) + d_\varpi(\prod_{i \geq 1} (1 + \theta_i \varpi^i), u_j \varpi^{-2n-1}) \end{aligned}$$

Now, $d_\varpi(\theta_0 \varpi^k, u_j \varpi^{-2n-1})$ is easy to compute:

$$\begin{aligned} d_\varpi(\theta_0 \varpi^k, u_j \varpi^{-2n-1}) &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \mathrm{res}_\varpi(u_j \varpi^{-2n-1} \theta_0^{-1} \varpi^{-k} \frac{d(\theta_0 \varpi^k)}{d\varpi}) \\ &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \mathrm{res}_\varpi(k u_j \varpi^{-2n-2}) \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_\varpi(\prod_{i \geq 1} (1 + \theta_i \varpi^i), u_j \varpi^{-2n-1}) &= \sum_{i \geq 1} d_\varpi(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1}) \\ &= \sum_{i=1}^{2n+1} d_\varpi(1 + \theta_i \varpi^i, u_j \varpi^{-2n-1}) \end{aligned}$$

since $d_\varpi(1+\theta_i \varpi^i, u_j \varpi^{-2n-1}) = 0$ if $i > 2n+1$ (see [8, p. 150], proof of Corollary). Moreover, by the same proof of Corollary in [8, p. 150], we have

$$\begin{aligned} d_\varpi(1 + \theta_{2n+1} \varpi^{2n+1}, u_j \varpi^{-2n-1}) &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}((2n+1) u_j \theta_{2n+1}) \\ &= \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(u_j \theta_{2n+1}). \end{aligned} \tag{4.7}$$

This last formula is a particular case of a more general formula we are about to prove.

In order to compute $d_\varpi(1+\theta_i\varpi^i, u_j\varpi^{-2n-1})$ for $i = 1, \ldots, 2n+1$, we need to find the Laurent series expansion of $(1+\theta_i\varpi^i)^{-1}$. This can be done by expanding the geometric series

$$(1+\theta_i\varpi^i)^{-1} = \sum_{j\geq 0}(-\theta_i\varpi^i)^j = 1-\theta_i\varpi i+\theta_i^2\varpi 2i-\theta_i^3\varpi 3i+\cdots$$

We have

$$u_j\varpi^{-2n-1}(1+\theta_i\varpi^i)^{-1}\frac{d}{d\varpi}(1+\theta_i\varpi^i) = \\ iu_j\theta_i\varpi^{-2n-1+i-1}(1-\theta_i\varpi^i+\theta_i^2\varpi^{2i}-\theta_i^3\varpi^{3i}+\cdots+(-1)^r\theta_i^r\varpi^{ri}+\cdots)$$

The residue will be nonzero if

$$-2n-1+i-1+ri = -1 \Leftrightarrow r = \frac{2n+1}{i}-1$$

Hence, $d_\varpi(1+\theta_i\varpi^i, u_j\varpi^{-2n-1}) = 0$ if $i \nmid 2n+1$. In particular, i must be odd.

We have:

$$d_\varpi(1+\theta_i\varpi^i, u_j\varpi^{-2n-1}) = \begin{cases} 0\ , & \text{if } i \nmid 2n+1 \\ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(u_j\theta_i^{(2n+1)/i})\ , & \text{if } i|2n+1 \end{cases}$$

In particular, we recover formula (4.7) by taking $i = 2n+1$.

From the above, we have established the following explicit formula.

Theorem 4.2. *Let K be a local function field of characteristic 2 with residue degree f, and let $\chi_{n,j}$ denote the quadratic character from (4.2). Then we have the explicit formula*

$$\chi_{n,j}(\alpha) = \sum_{i|2n+1}\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(u_j\theta_i^{(2n+1)/i})$$

where $\alpha = \varpi^k\theta_0\prod_{i\geq 1}(1+\theta_i\varpi^i) \in K^\times$, $n \geq 0$ and $j = 1, \ldots, f$.

For example, we have

$$\chi_{0,1}(\alpha) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}\,\theta_1, \quad \chi_{1,1}(\alpha) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\theta_1^3+\theta_3), \quad \chi_{2,1}(\alpha) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\theta_1^5+\theta_5),$$

where $\{1, u_2, \ldots, u_f\}$ is a basis of $\mathbb{F}_q/\mathbb{F}_2$.

4.2. *Ramification*

Quadratic extensions L/K are obtained by adjoining an $\mathbb{F}_2$-line $D \subset K/\wp(K)$. Therefore, $L = K(\wp^{-1}(D)) = K(\gamma)$ where $D = \mathrm{span}\{\beta+\wp(K)\}$, with $\gamma^2-\gamma = \beta$. In particular, if $\beta_0 \in \mathfrak{o}\backslash\mathfrak{p}$ such that the image of β_0 in $\mathfrak{o}/\mathfrak{p}$ has nonzero trace in $\mathbb{F}_2$, the $\mathbb{F}_2$-line $V_0 = \mathrm{span}\{\beta_0+\wp(K)\}$ contains all the cosets $\beta_i+\wp(K)$ where β_i is an integer and so $K(\wp^{-1}(\mathfrak{o})) = K(\wp^{-1}(V_0)) = K(\gamma_0)$ where $\gamma_0^2-\gamma_0 = \beta_0$ gives the unramified quadratic extension, see [6, Proposition 12, p. 412].

Biquadratic extensions are computed the same way, by considering $\mathbb{F}_2$-planes $W = \mathrm{span}\{\beta_1+\wp(K), \beta_2+\wp(K)\} \subset K/\wp(K)$. Therefore, if $\beta_1+\wp(K)$ and $\beta_2+\wp(K)$ are $\mathbb{F}_2$-linearly independent then $K(\wp^{-1}(W)) := K(\gamma_1,\gamma_2)$ is biquadratic, where $\gamma_1^2-\gamma_1 = \beta_1$ and $\gamma_2^2-\gamma_2 = \beta_2$, $\gamma_1,\gamma_2 \in K^s$. Therefore, $K(\gamma_1,\gamma_2)/K$ is biquadratic if $\beta_2-\beta_1 \notin \wp(K)$.

A biquadratic extension containing the line V_0 is of the form $K(\gamma_0,\gamma)/K$. There are countably many biquadratic extensions L_0/K containing the unramified quadratic extension. They have ramification index $e(L_0/K) = 2$. And there are countably many biquadratic extensions L/K which do not contain the unramified quadratic extension. They have ramification index $e(L/K) = 4$.

So, there is a plentiful supply of biquadratic extensions $K(\gamma_1, \gamma_2)/K$.

The space $K/\wp(K)$ comes with a filtration

$$0 \subset_1 V_0 \subset_f V_1 = V_2 \subset_f V_3 = V_4 \subset_f \cdots \subset K/\wp(K) \tag{4.8}$$

where V_0 is the image of $\mathfrak{o}_K$ and V_i $(i > 0)$ is the image of $\mathfrak{p}^{-i}$ under the canonical surjection $K \to K/\wp(K)$. For $K = \mathbb{F}_q((\varpi))$ and $i > 0$, each inclusion $V_{2i} \subset_f V_{2i+1}$ is a sub-$\mathbb{F}_2$-space of codimension f. The $\mathbb{F}_2$-dimension of V_n is

$$\dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil f, \tag{4.9}$$

for every $n \in \mathbb{N}$, where $\lceil x \rceil$ is the smallest integer bigger than x.

Let L/K denote a Galois extension with Galois group G. For each $i \geq -1$ we define the i^{th}-ramification subgroup of G (in the lower numbering) to be:

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{p}_L^{i+1}, \forall x \in \mathfrak{o}_L\}.$$

An integer t is a *break* for the filtration $\{G_i\}_{i \geq -1}$ if $G_t \neq G_{t+1}$. The study of ramification groups $\{G_i\}_{i \geq -1}$ is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering $\{G^i\}_{i \geq -1}$ and defined by the *Hasse-Herbrand function* $\psi = \psi_{L/K}$:

$$G^u = G_{\psi(u)}.$$

In particular, $G^{-1} = G_{-1} = G$ and $G^0 = G_0$, since $\psi(-1) = -1$ and $\psi(0) = 0$.

Now, in analogy with the lower notation, a real number $t \geq -1$ is a *break* for the filtration $\{G^i\}_{i \geq -1}$ if

$$\forall \varepsilon > 0, \, G^t \neq G^{t+\varepsilon}. \tag{4.10}$$

We define

$$G^{t+} := \bigcap_{r > t} G^r. \tag{4.11}$$

Then t is a break of the filtration if and only if $G^{t+} \neq G^t$. The set of breaks of the filtration is countably infinite and need not consist of integers.

If G is abelian, it follows from Hasse-Arf theorem [8, p.91] that the breaks are integers and (4.10) is equivalent to

$$G^t \neq G^{t+1}.$$

Let $G_2 = \mathrm{Gal}(K_2/K)$ be the Galois group of the maximal abelian extension of exponent 2, $K_2 = K(\wp^{-1}(K))$. Since $G_2 \cong K^\times/K^{\times 2}$, the nondegenerate pairing (4.1) coincides with the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

The profinite group G_2 comes equipped with a ramification filtration $(G_2^u)_{u \geq -1}$ in the upper numbering, see [6, p.409]. For $u \geq 0$, we have an orthogonal relation [6, Proposition 17, p.415]

$$(G_2^u)^\perp = \overline{\mathfrak{p}^{-\lceil u \rceil + 1}} = V_{\lceil u \rceil - 1} \tag{4.12}$$

under the pairing $G_2 \times K/\wp(K) \to \mathbb{Z}/2\mathbb{Z}$.

Since the upper filtration is more suitable for quotients, we will compute the upper breaks. By using the Hasse-Herbrand function it is then possible to compute the lower breaks in order to obtain the lower ramification filtration.

According to [6, Proposition 17], the positive breaks in the filtration $(G^v)_v$ occur precisely at integers prime to p. So, for $ch(K) = 2$, the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If G is cyclic of prime order, then there is a unique break for any decreasing filtration $(G^v)_v$ (see [6], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane $W \subset K/\wp(K)$, the filtration (4.8) $(V_i)_i$ on $K/\wp(K)$ induces a filtration $(W_i)_i$ on W, where $W_i = W \cap V_i$. There are three possibilities for the filtration breaks on a plane and we will consider each case individually.

Case 1: W contains the line V_0, i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of K. The extension has residue degree $f(L_0/K) = 2$ and ramification index $e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration $(W_i)_i$ looks like

$$0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.$$

By the orthogonality relation (4.12), the upper ramification filtration on $G = \mathrm{Gal}(L_0/K)$ looks like

$$\{1\} = \cdots = G^{t+1} \subset_1 G^t = \cdots = G^0 \subset_1 G^{-1} = G$$

Therefore, the upper ramification breaks occur at -1 and t.

The number of such W is equal to the number of planes in V_t containing the line V_0 but not contained in the subspace V_{t-1}. This number can be computed and equals the number of biquadratic extensions of K containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1, t)$, $t > 0$ and odd. Here is an example.

Example 4.3. *The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks* $(-1, 1)$ *is equal to the number of planes in an* $1 + f$*-dimensional* $\mathbb{F}_2$*-space, containing the line* V_0*. There are precisely*

$$1 + 2 + 2^2 + \cdots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1$$

of such biquadratic extensions.

Case 2.1: W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t-1} \subset_2 W_t = W$$

for some integer t, necessarily odd.

The number of such W is equal to the number of planes in V_t whose intersection with V_{t-1} is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((\varpi))$, **case 2.1** does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification filtration on $G = \mathrm{Gal}(L/K)$ looks like

$$\{1\} = \cdots = G^{t+1} \subset_2 G^t = \cdots = G^{-1} = G$$

Therefore, there is a single upper ramification break occurring at $t > 0$ and is necessarily odd.

For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes W contained in an $\mathbb{F}_2$-space of dimension $1 + fi$, $t = 2i - 1$, which are transverse to a given codimension-f $\mathbb{F}_2$-space.

Case 2.2: W does not contains the line V_0 and the induced filtration on the plane W looks like

$$0 = W_{t_1-1} \subset_1 W_{t_1} = W_{t_2-1} \subset_1 W_{t_2} = W$$

for some integers t_1 and t_2, necessarily odd, with $0 < t_1 < t_2$.

The orthogonality relation for this case implies that the upper ramification filtration on $G = \mathrm{Gal}(L/K)$ looks like

$$\{1\} = \cdots = G^{t_2+1} \subset_1 G^{t_2} = \cdots = G^{t_1+1} \subset_1 G^{t_1} = \cdots = G$$

The upper ramification breaks occur at odd integers t_1 and t_2.

There is only a finite number of such biquadratic extensions, for a given pair of upper breaks (t_1, t_2).

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