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Control of chaotic dynamics in an OLG economic model

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Abstract. This paper deals with the control of chaotic economic motion. We show that very complicated dynamics arising, e.g., from an overlapping generations model (OLG) with production and an endogenous intertemporal decision between labour and leisure, which produces chaos, can in fact be controlled with relative simplicity. The aperiodic and very complicated motion that stems from this model can be subject to control by small perturbations in its parameters and turned into a stable steady state or into a regular cycle. Therefore, the system can be controlled without changing of its original properties. To perform the control of the totally unstable equilibrium (both eigenvalues with modulus greater than unity) in this economic model we apply the pole-placement technique, developed by Romeiras, Grebogi, Ott and Dayawansa (1992).

The application of control methods to chaotic economic dynamics may raise serious reservations, at least on mathematical and logical grounds, to some recent views on economics which have argued that economic policy becomes useless in the presence of chaotic motion (and thus, that the performance of the economic system cannot be improved by public intervention, i.e., that the amplitude of cycles can not be controlled or reduced). In fact, the fine tuning of the system (that is, the control) can be performed without having to rely only on infinitesimal accuracy in the perturbation to the system, because the control can be performed with larger or smaller perturbations, but neither too large (because these would lead to a different fixed point of the system, therefore modifying its original nature), nor too small because the control becomes too inefficient.

1. Introduction

"All stable processes, we shall predict. All unstable processes, we shall control." John von Neumann, circa 1950 [1]

In a very famous book, James Gleick (1988) portrayed a fantastic view of a new set of mathematical results that is nowadays referred to as "chaos theory". The book is in fact written in a very elegant, and even eloquent, prose and this has certainly helped to turn the book into a best seller and to convert the "idea" of chaos into a popular one in many academic fields (economics included) and in general intellectual discussions, even outside the core of scientific

[1] From Dyson (1988, 182).

research. The view of chaos that has become popular seems to be one crystallized by the famous allegory of the butterfly effect, in which a system that follows a chaotic motion is a process doomed to be *totally out of order, totally out of control* and, therefore, *completely unpredictable.* Therefore, according to this view the scientific approach described above in simple but powerful words by one of the most brilliant mathematicians and intellectual minds of the previous century (John von Neumann) apparently seems just a mere relic of past scientific glory. As Gleick's forcefully puts it: "where chaos begins, classical science stops". In fact, this view of chaos is not confined to science writers outside academia. As Freeman Dyson expressed in "Infinite in All Directions", von Neumann's vision was wrong because:

> "A chaotic motion is generally neither predictable nor controllable. It is unpredictable because a small disturbance will produce exponentially growing perturbation of the motion. It is uncontrollable because small disturbances lead only to other chaotic motions and not to any stable and predictable alternative. *Von Neumann's mistake was to imagine that every unstable motion could be nudged into a stable motion by small pushes and pulls applied at the right places. The same mistake is still frequently made by economists and social planners, not to mention Marxist historians*" (1988, 183, emphasis added)

Therefore, it is not surprising that in the field of economics there is also a widespread feeling among many economists that the advent of chaos may lead to a revolution in economic thought in both the academic world and in policy–making institutions. The advent of chaos and an alledgedly new scientific revolution have captured the imagination of many economists and, e.g., Paul Ormerod, a well established economist, journalist, and entrepreneur had apparently no doubts to proclaim throughout the 1990s the "Death of Economics" in two largely read books (1995, 1999). The implications would be concentrated on two major points. *Firstly*, if modern economies are properly described as moving according to chaotic motion, then they seem to be almost impossible to understand, to predict, and to control using conventional analytical methods (i.e, formal mathematical models). This would render economic theory and policy, as they have been done and practiced over the previous decades, totally irrelevant in contemporary economies. *Secondly*, as a corollary, any improvement on the functioning of these economies would require a radical change to their basic structures, because the crises and booms associated with the dynamics of capitalist structures, by being chaotic manifestations, can be neither controllable nor predictable. A view expressed by Medio in the late 1980s shows these points very clearly:

> "The existence of chaos in any specific model ... may have vast consequences both for economic theory and policy. Just to mention two of them: if irregular fluctuations depend on the structure of the system, rather than on external disturbances, *intervention to eliminate or reduce them will have to change the system* rather than shield it from the shocks. Also if the behavior of the system is extremely sensitive to changes in initial conditions (and therefore to shocks), as it is known to be the case in most types of chaos, *effective 'fine tuning' becomes impossible, unless policy measures are infinitely accurate."* (1987, 336, emphasis added)

While the arguments above may have been acceptable prior to the early 1990s given the existing state of knowledge about chaotic systems at that time, they may well look as possibly misleading statements by now. In the last ten years or so there has been a remarkable amount of new results in the analysis of chaotic systems and these can not be overlooked. For example, control, targeting, synchronization, and forecasting of chaotic motion have provided well established results in the fields of applied mathematics, physics and engineering that lead us to question the validity of such arguments at least on the purely logical or mathematical

grounds.[2]

In this paper we apply one of these new techniques — the control of chaos — to show that the presence of chaotic motion in economic processes does not necessarily need to be interpreted as a curse for economic theory and economic policy, for three major reasons:

- Firstly, instead of rendering economic policy impossible or useless, the existence of endogenous chaotic cycles may provide a justification for public policy even in cases where conventional economic theory would sustain a no–intervention verdict;

- Secondly, in order to control economic chaotic motion we do not need to change the fundamental characteristics of the system, we just have to impose upon the dynamics some small (or tiny) perturbations, an approach that frequently contrasts with the control of non linear non–chaotic dynamical systems.

- Thirdly, the fine tuning of the system (that is, the control) can in fact be performed without having to rely *only* on infinitesimal accuracy in the perturbation process, because the control can be performed with larger or smaller perturbations, but neither too large (because these would lead to a different fixed point of the system, therefore modifying its original nature), nor too small because the control becomes too inefficient.

Take the example of an overlapping generations model with no bequests, and no taxation.[3] The introduction into this model of a constant relative risk aversion utility function and a linear Leontief technology leads to chaotic motion, and to endogenous business cycles of large amplitude as we shall see. The application of a small external perturbation to one accessible parameter of the model leaves the fundamental characteristics of the system unchanged — the fixed point that forms the basis of attraction remains the same — turns the fixed point from an unstable into a stable one, and eliminates those large business cycles (in other cases their amplitude may be reduced). As we will show, this control fulfills the points above referred to, and this happens even in models which exhibit totally unstable equilibria (both eigenvalues have modulus greater than unity) or even hyperchaos (both Lyapunov exponents higher than zero).

The control method that we are going to apply is a well known feedback control technique initially developed by Romeiras, Grebogi, Ott and Dayawansa (RGOD) for chaos control (1992). They made the very important observation that a chaotic attractor has embedded within it a dense set of unstable periodic orbits. Since they wish to make only very small perturbations to the system they do not envision creating new orbits with different properties from the existing ones. Thus, they seek to exploit the dynamics of the already existing unstable periodic orbits. This method uses a linear approximation to the dynamics in the neighborhood of the desired periodic orbit, and consists in producing small perturbations to a system–wide accessible parameter to stabilize all unstable directions.

We are not aware of many papers in economics dealing with the process of controlling chaotic models despite an already impressive amount of work on the modelling side of economic chaotic dynamics.[4] Those which we have come across include Holyst *et al.* (1996) and Kopel (1997) who studied chaotic processes of dynamic games of two oligopolistic firms in a partial equilibrium framework, Kaas (1998) who applied control to a non–optimal conventional macroeconomic model, and Bala *et al.* (1998) who control chaos arising in the context of a tatonnement process

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[2] We should stress that the above sentence by Medio dates back to 1987. It is perfectly possible that Medio has now a different position concerning the powers of chaos control. However, many other economists have used his authority in the field to argue in favour of the irrelevance of economic policy due to chaotic motion.

[3] Apart from the fact that transfer or bequests across generations would improve social welfare (which would vindicate public intervention), there is no other theoretical reason to justify the intervention of public agencies in the economic process in this model.

[4] For excellent surveys see, e.g., Benhabib (1992), Day (1999), Medio and Gallo (1995), Brock and Hommes (1997), Barnett *et al.* (1999), Lorenz (1993), and Boldrin *et al.* (2000).

of exchange economies. All these papers perform the control of chaotic motion using the OGY [Ott, Grebogi and Yorke (1990)] method which is appropriate for the control of saddle point instability or the Pyragas time-delay feedback control method [Pyragas (1992)].

This paper is organized as follows. In section 2 we present the basic characteristics of the overlapping generations model (OLG) with production and optimal leisure choice, a model that has been extensively studied as a source of generating chaotic motion. The model will have two basic characteristics: constant relative risk aversion in utility and a Leontief technology. In section 3, the dynamics of the OLG model is studied with some detail, including stable and unstable fixed points, periodic and quase-periodic motion, bifurcations, Lyapunov exponents, correlation dimensions and chaos. Section 4 deals with the control of chaotic motion, and the process of control is achieved using a relative risk aversion coefficient. Section 5 concludes.

2. An Overlapping Generations Model

Medio and Negroni (1996) used the basic OLG framework to study various combinations of utility and production functions that would lead to chaotic behavior. These combinations include CES *versus* Leontief (or fixed factor proportions) production functions and constant absolute *versus* constant relative risk aversion utility functions. In this paper we use a combination of a constant relative risk aversion (CRRA) utility function with a Leontief technology (L) — from where stems the shortname of the model as CRRAL — to show that chaotic economic dynamics can be easily subject to control. All other possible combinations of utility and production functions in the paper by Medio and Negroni also lead to chaos. However, as in the control of chaotic dynamics each specific dynamics may require a particular technique, we can only discuss in this paper one of those combinations due to space shortage. For example, in Mendes and Mendes (2001) we apply a different technique to control the chaotic dynamics that arises from a constant absolute risk aversion with a Leontief technology (CARAL economy), which is based on the OGY method and in Mendes and Mendes (2005) we apply the technique presented in Yang *et al.* (2000) in order to control hyperchaos in a CRRAL model.

We consider an overlapping generations model with production, where economic agents live for two periods (young at t, and old age at $t+1$), and in which there is an optimal intertemporal choice between labour and leisure: they work only in the first period and they consume in both periods. We also consider that there exists a unique commodity in this economy which can be either consumed or used in the production process as investment. Therefore, this economy has two major agents: firms that produce goods and services by hiring labor and capital services and maximize profits, and families which maximize utility and rent labour services in exchange for a wage rate.

Utility side. We will use the following designations: w_t for the real wage rate, R_{t+1} as the gross real interest rate, $u_1(c_t)$ is the utility of consumption in the first period, $u_2(c_{t+1})$ the utility of consumption in the second period, $v(l_t)$ the disutility of labour in the first period, s_t as the level of savings per person in the first period. We assume that the functions u_1, u_2, v are continuous and monotonously increasing on $\mathbb{R}_+$, with u_1, u_2 concave and v convex on $\mathbb{R}_+$.

The dynamic optimization problem can be written as

$$\max. \quad u_1(c_t) + u_2(c_{t+1}) - v(l_t) \tag{1}$$

$$s.t.$$
$$s_t \leq w_t l_t - c_t$$
$$c_{t+1} \leq R_{t+1} s_t$$
$$c_t, c_{t+1}, s_t, l_t > 0$$

Setting the Lagrangean for this optimal problem[5] we obtain

$$\mathcal{L} = u_1\left(c_t\right) + u_2\left(c_{t+1}\right) - v\left(l_t\right) + \lambda\left[\left(w_t l_t - c_t\right) R_{t+1} - c_{t+1}\right].$$

The optimal problem for each family can be determined by the first order conditions (FOCs) with respect to the three decision variables (c_t, c_{t+1}, l_t) and the multiplier (λ)

$$u_1'\left(c_t\right) - \lambda R_{t+1} = 0 \tag{2}$$
$$u_2'\left(c_{t+1}\right) - \lambda = 0 \tag{3}$$
$$v'\left(l_t\right) + \lambda w R_{t+1} = 0 \tag{4}$$
$$\left(w_t l_t - c_t\right) R_{t+1} - c_{t+1} = 0 \tag{5}$$

We shall proceed as follows to simplify this problem. Firstly, use equations (4) and (5) to eliminate w_t, and then obtain from the two first FOCs the result $R_{t+1} = u_1'\left(c_t\right) / u_2'\left(c_{t+1}\right)$. Finally, substitute this result for R_{t+1} into the first step, and the optimal rule for intertemporal consumption and leisure over time will appear as

$$u_1'\left(c_t\right) c_t + u_2'\left(c_{t+1}\right) c_{t+1} - v'\left(l_t\right) l_t = 0. \tag{6}$$

Assuming constant relative risk aversion in all utility functions

$$u_1\left(c_t\right) = \frac{1}{\theta} c_t^\theta \qquad 0 < \theta < 1$$
$$u_2\left(c_{t+1}\right) = \frac{1}{\alpha} c_{t+1}^\alpha, \quad 0 < \alpha < 1 \tag{7}$$
$$v\left(l_t\right) = \frac{1}{\gamma} l_t^\gamma, \qquad \gamma > 1$$

the maximization of intertemporal utility in this CRRA framework leads to

$$c_{t+1} = \left(l_t^\gamma - c_t^\theta\right)^{1/\alpha}. \tag{8}$$

This is the first fundamental equation that characterizes the dynamics of this model, and represents the optimal evolution of consumption, derived from the consumer's intertemporal choice of consumption and leisure.

Technological side. To obtain the second equation of our dynamic system, we have to look at the technological side of the economy. We consider a linear Leontief production technology

$$y_t = \min[a l_t, b k_{t-1}]. \tag{9}$$

This equation assumes that output per person in period t, (y_t), is obtained by a linear combination of the amount of labour allocated to production in period t, (l_t), and the volume of capital accumulated in $t-1$, (k_{t-1}). The two parameters satisfy the following constraints: $a = 1$ for simplicity, and $b > 1$ for viability of capital accumulation.

The assumption of full employment and the restriction $a = 1$ lead to the result $y_t = l_t$. Moreover, the assumption of a constant capital/output ratio leads to $y_t = b k_{t-1}$. Taking into account the equilibrium condition in the product market, $y_t = k_t + c_t$, we obtain

$$y_t = b\left(y_{t-1} - c_{t-1}\right). \tag{10}$$

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[5] Note that the two constraints can in fact be reduced to only one by cancelling s_t.

Moving forward one period, and using the result $y_t = l_t$, we can obtain the second fundamental equation that characterizes the dynamics of this OLG model

$$l_{t+1} = b\left(l_t - c_t\right), \text{ with } b > 1. \tag{11}$$

Equations (8) and (11) represent the evolution of the system that is compatible with intertemporal optimization in constant relative risk aversion utility and equilibrium conditions in a Leontief economy.

3. The Dynamics of the CRRAL Economy

We have the following nonlinear 2-dimensional map which characterizes the overlapping generation model of a CRRAL economy

$$\begin{cases} c_{t+1} = \left(l_t^\gamma - c_t^\theta\right)^{1/\alpha} \\ \\ l_{t+1} = b\left(l_t - c_t\right) \end{cases} \tag{12}$$

where $\gamma > 1$, $b > 1$, $0 < \alpha, \theta < 1$ are the parameters of the system. Despite its apparent simple form, the map presents an extremely complicated dynamic behavior. Different routes to chaos and lack of explicit analytical expressions for equilibria are noted for this map.

To compute the fixed points we have to solve the nonlinear system given by

$$\begin{cases} \left(l_t^\gamma - c_t^\theta\right)^{1/\alpha} = c_t \\ \\ b\left(l_t - c_t\right) = l_t. \end{cases} \tag{13}$$

There exists two fixed points: the first one is the trivial $E_1 = (0,0)$, which is always locally unstable, and the second one is $E_2 = (c_* > 0, l_* > 1)$ which we cannot compute explicitly but it was showed by Medio and Negroni (1996) [6] that is strictly positive. The equilibrium E_2 it is stable if satisfies the following conditions

$$\begin{cases} 1 + Tr\left(J\left(E_2\right)\right) + Det\left(J\left(E_2\right)\right) > 0 \\ 1 - Tr\left(J\left(E_2\right)\right) + Det\left(J\left(E_2\right)\right) > 0 \\ 1 - Det\left(J\left(E_2\right)\right) > 0 \end{cases} \tag{14}$$

where $J\left(E_2\right)$ is the Jacobian matrix computed at the fixed point E_2

$$J\left(E_2\right) = \begin{bmatrix} -\dfrac{\theta}{\alpha}c^{\theta-\alpha} & \dfrac{\gamma}{\alpha}\left(\dfrac{b}{b-1}\right)^{\gamma-1}c^{\gamma-\alpha} \\ -b & b \end{bmatrix} \tag{15}$$

and $Tr\left(J\left(E_2\right)\right)$ is the trace of the Jacobian matrix. This is a well known sufficient condition for the local stability of an equilibrium and is giving necessary and sufficient conditions for the two roots $\lambda_{1,2}$ of the characteristic equation to be inside the unit circle of the complex plane. Since

$$Tr\left(J\left(E_2\right)\right) = -\frac{\theta}{\alpha}c^{\theta-\alpha} + b$$
$$Det\left(J\left(E_2\right)\right) = (b-1)\frac{\gamma}{\alpha} + [\gamma\left(b-1\right) - b\theta]\frac{c^{\theta-\alpha}}{\alpha} \tag{16}$$

[6] In the paper of Medio and Negroni, the fixed point E_2 is defined for $(c_* > 1, l_* > 1)$, when actually almost all of the values of the c coordinate computed here lie in the interval $]0, 1[$.
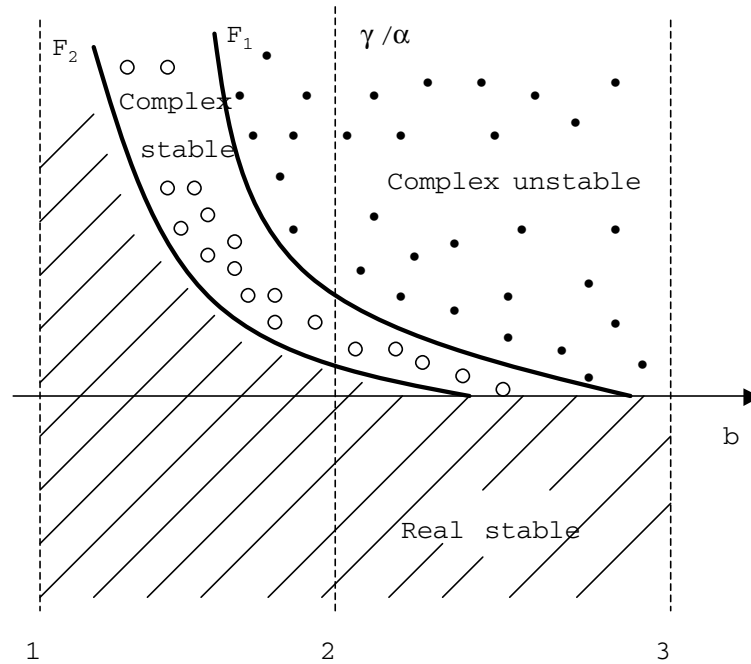
Figure 1. Stability of the equilibrium E_2.

the stability conditions (14) imply that the fixed point E_2 is stable if γ is sufficiently small, α is sufficiently large, or b is sufficiently small. In *Figure 1* we show the stability areas associated with the equilibrium E_2 where the curve F_1 denotes the condition $1 - Det\left(J\left(E_2\right)\right) = 0$ from relation (14), that is $\gamma/\alpha = (1+b)(2(b-1))^{-1}$, and F_2 denotes the discriminant expression $(Tr\left(J\left(E_2\right)\right))^2 - 4Det\left(J\left(E_2\right)\right) = 0$, that is $\gamma/\alpha = (1+b)^2(8(b-1))^{-1}$.

The purpose of this paper is to control chaotic orbits and, therefore, we should only be interested in the values of the parameters for which the map shows chaotic behavior. We will choose the parameter's values such that they are located somewhere in the black bullets region in *Figure 1* (complex conjugate eigenvalues, unstable equilibrium), that is for $\gamma/\alpha > (1+b)(2(b-1))^{-1}$.

As we are dealing with a nonlinear 2–dimensional map, and as the theoretical tools to prove the existence of chaotic motion in 2–dimensions are still very poor, we do this resorting to computer numerical approximations. Without loss of generality we fix $\alpha = \theta = 0.2, b = 1.2$ and assume that γ can vary. We consider the following initial conditions $(c_0, l_0) = (0.1, 1.16)$ situated in the basin of attraction of the fixed point E_2 and we start to study the behavior of the system when the parameter γ is varied in the interval $]1, 1.73[$.

Generically, when the control parameter varies, a periodic solution can lose stability through various types of bifurcations and the resulting orbit depends on how the multipliers leave the unit circle. Recalling *Figure 1*, we realize that starting from a stable configuration and increasing γ we have to pass through the boundaries delimited by F_1 and F_2, and so, loss of stability (boundary F_1) takes place through a Neimark-Sacker bifurcation. The Neimark-Sacker bifurcation (or secondary Hopf bifurcation) is a local bifurcation which produces a qualitative change in some neighborhood of the fixed point when a pair of complex conjugate eigenvalues leave the unit circle away from the real axis and, as a consequence, an invariant closed curve (circle) is bifurcating for some value of γ, say γ_*, around the fixed point. This (unique) invariant circle occurs if certain, rather general, nonresonance conditions hold for the normal form of the system. We assume

that these conditions [7] are satisfied here and for a rigorous proof of existence of Neimark-Sacker bifurcations for generic 2-dimensional maps, we refer to Kuznetsov (1998).

Along this process we numerically encounter a multitorus (or torus breakdown) route to chaos. In this route to chaos, a torus attractor bifurcates into periodic orbits of consecutively increasing (decreasing) periods, i.e., windows of quasi periodic and periodic behavior appear alternately as the parameter γ is changed. After several such bifurcations, a periodic orbit finally bifurcates into a chaotic attractor. These various results will be discussed and illustrated by a numerous set of figures in the remaining part of this section.

The first Neimark-Sacker bifurcation takes place for $\gamma = 1.10$ and for values lower than $\gamma = 1.10$ a stable equilibrium exists. We start with an initial value of the control parameter, let us say $\gamma = 1.098$. As the analysis above indicates this value leads to a stable equilibrium point, and the attractor of this type of equilibrium is ilustrated in *Figure 2*. Both variables of the dynamic system (c_t and l_t, respectively, consumption and labour services per person) converge towards a unique and stable point independently from the initial state of the economy. The equilibrium point is characterized by the optimal intertemporal values $c_* = 0.2419$, $l_* = 1.4518$. The eigenvalues of the Jacobian matrix computed at the equilibrium point are $\lambda_{1,2} = 0.0999 \pm 0.9929i$ with $|\lambda_{1,2}| \cong 0.9960$. This dynamic process can be also represented by the time series of both variables, which are also shown in *Figure 2*. As we can easily see both converge steadily and cyclically towards their steady state values.

As we have already mentioned the Neimark-Sacker bifurcation takes place for $\gamma = 1.10$, bifurcation value obtained from the stability condition $\gamma/\alpha = (1+b)(2(b-1))^{-1}$. For this parameter value, the equilibrium point occurs at $c_* = 0.2417$, $l_* = 1.4506$ and the associated pair of complex conjugate eigenvalues are $\lambda = 0.1000 \pm 0.9949i$ with $|\lambda| \cong 1.0000$, which shows that varying the parameter γ from 1.098 to 1.10, the eigenvalues are approximating the unit circle and the equilibrium is changing its stability properties through a Neimark-Sacker bifurcation. *Figure 3* illustrates the phase plot and the coordinates time series for the bifurcation value of γ.

Continuing to increase the value of γ, we see what happens for $\gamma = 1.11$. The coordinates of the equilibrium are $c_* = 0.2407$, $l_* = 1.4447$ and the associated eigenvalues are $\lambda = 0.0999 \pm 1.0049i$. The modulus of the complex conjugate eigenvalues is $|\lambda| \cong 1.020$, and so we can conclude that the equilibrium became unstable and an invariant closed curve was created around the fixed point, which is showed in the *Figure 4* together with the time series of the two state variables in this control problem. As one would expect from a quasi periodic motion, the phase plot is in the form of a smooth closed curve (the cross section of a two-torus).

As γ is further increased, however, the phase plot starts to fold and — interrupted by periodic windows — a quasi periodic transition to chaos takes place. We can see that the "circle" after being stretched, shrunk and folded creates a new phenomena: the breakdown of the invariant closed curve (see for instance *Figure 1*), which lead to the appearance of various invariant closed curves. Looking at the time series of the state coordinates, we can observe an expansion in the windows associated with each one of the two sequences.

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[7] A Neimark-Sacker bifurcation or Hopf bifurcation for maps is characterized by the following: given a parameter dependent map on $\mathbb{R}^n, n \geq 2$, $x \longmapsto A(a)x + G(x,a)$ with

(N1) G is a C^k-smooth mapping, $k \geq 2$, from $\mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}^n$, $G(0,a) = 0, G_x(0,a) = 0, a \in \mathbb{R}$

(N2) $A(a^*)$ has a complex conjugate pair of eigenvalues with modulus 1, i.e., $|\lambda_{1,2}| = 1$, while all other eigenvalues have modulus strictly less than one

(N3) $r'(a^*) \neq 0$, where $r(a)$ is the modulus of the branch of eigenvalues with $r(a^*) = 1$.

Under the above hypotheses the map has an invariant closed curve of radius $O\left(\sqrt{|a^* - a|}\right)$ surrounding the origin for all a in a one-side neighborhood of a^*. The closed curve is attracting (repelling) if zero is an asymptotically stable (unstable) fixed point of the map at $a = a^*$.
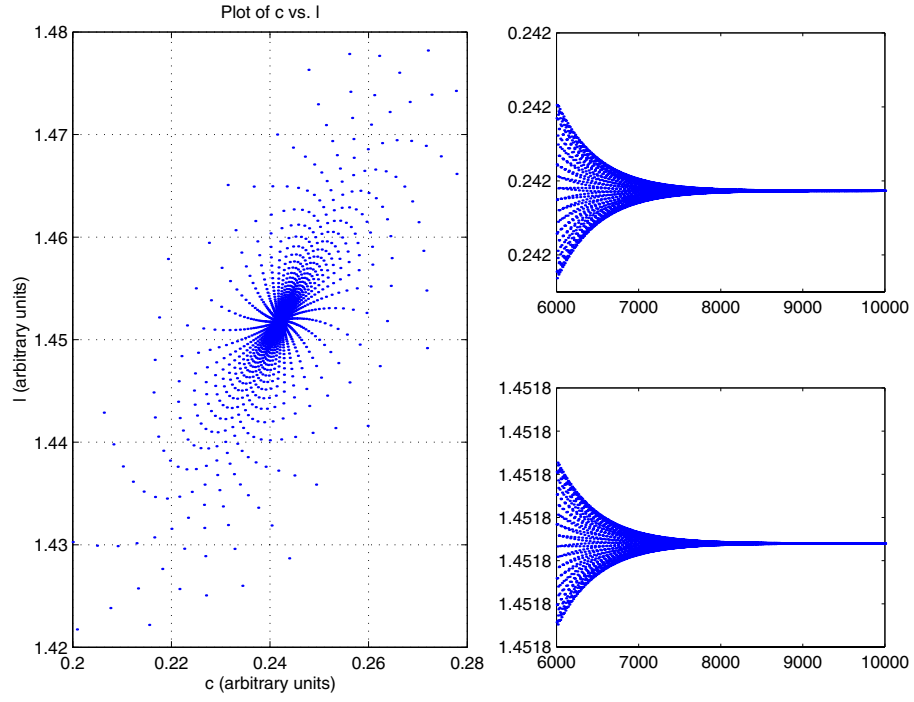
Figure 2. $\gamma = 1.098$, the stable fixed point before the first Neimark-Sacker bifurcation.
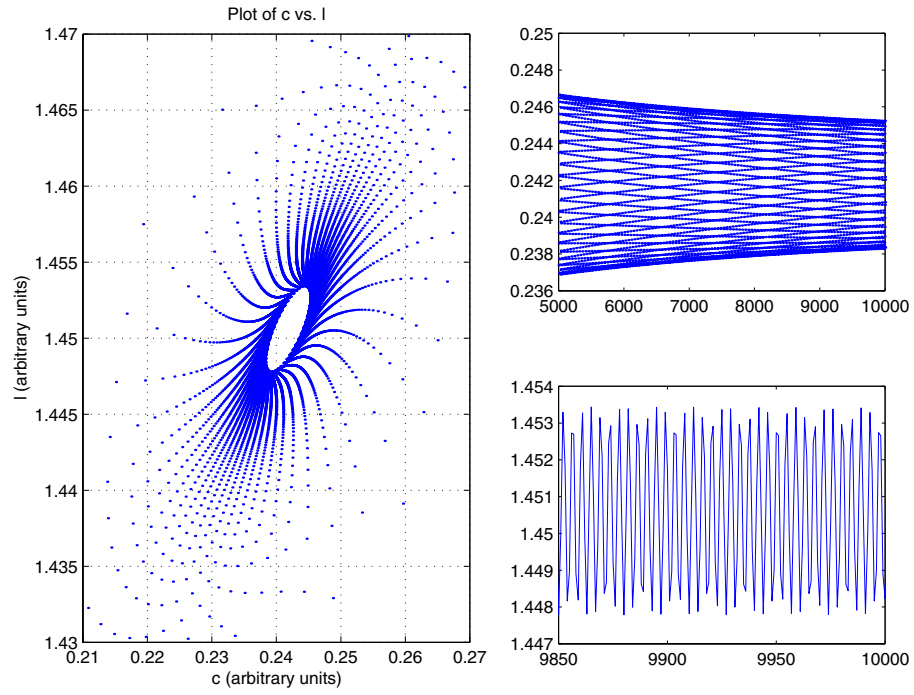

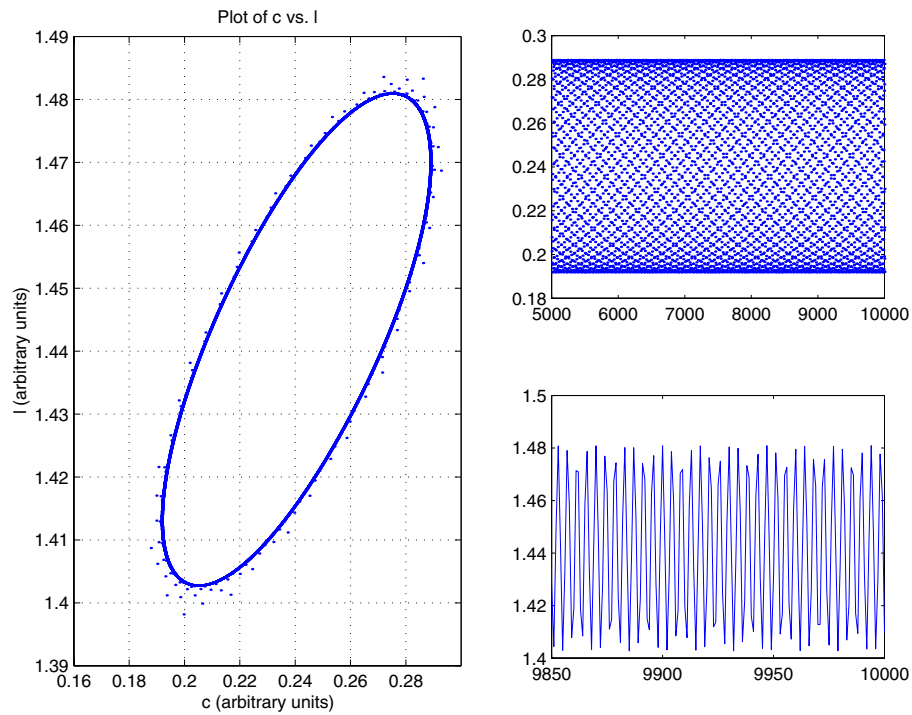**Figure 3.** $\gamma = 1.10$, the first Neimark-Sacker bifurcation.

Figure 4. $\gamma = 1.11$, the stable invariant closed curve around the fixed point created after the bifurcation.
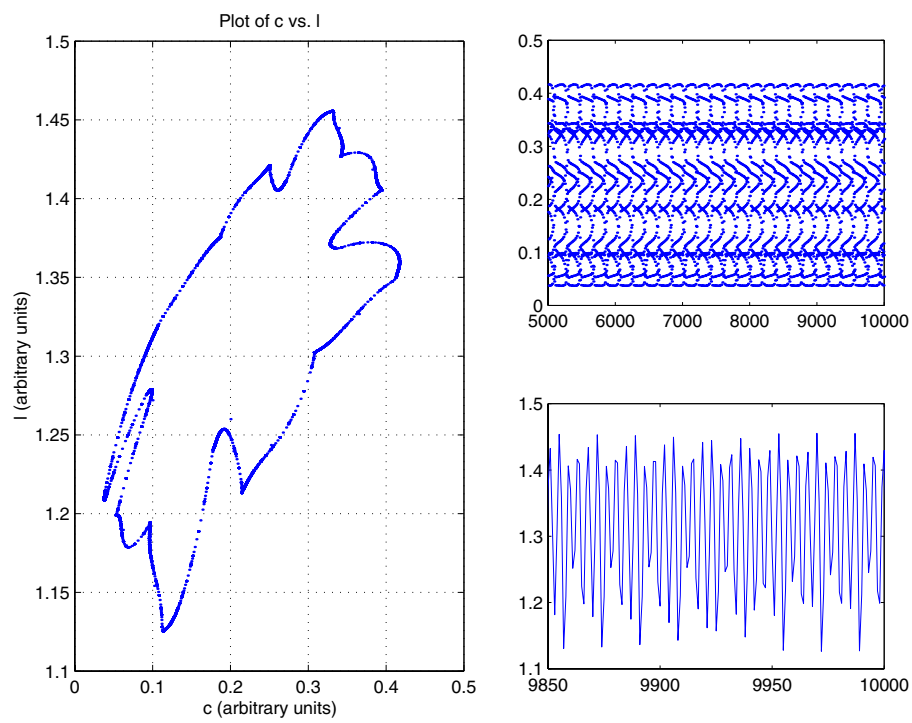

Figure 5. $\gamma = 1.316$, the breakdown of the invariant closed curve.
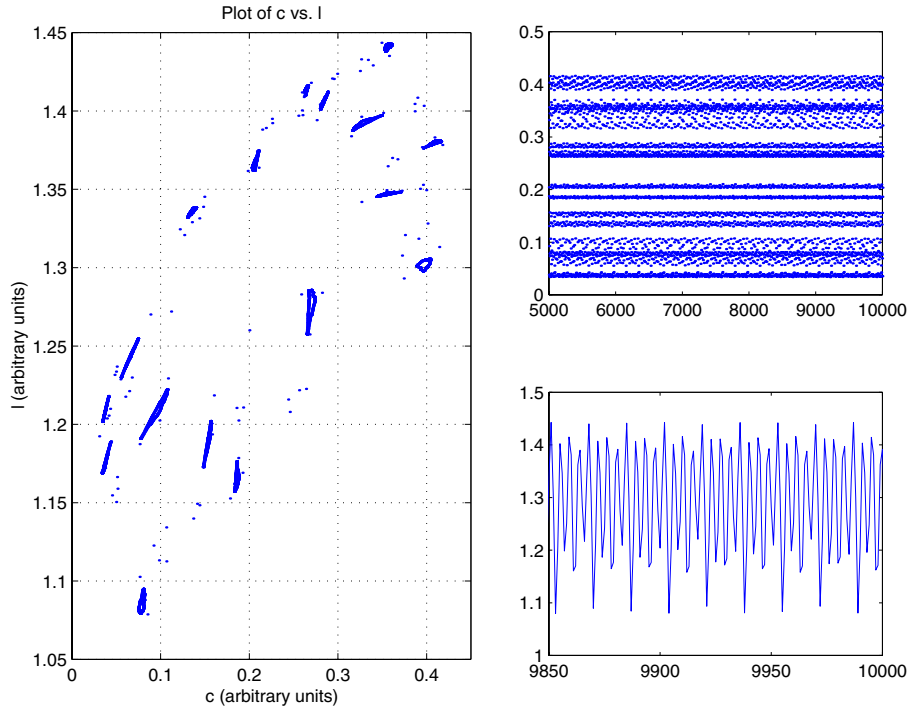
Figure 6. $\gamma = 1.362$, the existence of multiple invariant closed curves.

For $\gamma > 1.32$ we obtain multiple invariant closed curves brought by Neimark-Sacker bifurcations of iterates of the original map. In these cases, the dynamics from one circle to another are periodic (and thus easily predictable), but the dynamics on each closed curve, may be periodic or quasi periodic. Moreover, these closed curves may break, leading to multiple fractal tori on which the dynamics are chaotic. Following the Neimark-Sacker bifurcations, quasiperiodic solutions with windows of frequency locking appear. The radius of the quasiperiodic solution grows as γ is further increased. *Figure 6* represents seventeen invariant closed curves brought about by a Neimark-Sacker bifurcation of the 17th iterate of the map of the system, obtained for $\gamma = 1.362$. Following our procedure, we also present the time series associated with each of the coordinates which shows very clearly the quasi periodicity of the orbits.

A strange attractor is produced by successive stretching and folding. The attractor in *Figure 7* is a bounded region in the phase space to which all sufficiently close trajectories are asymptotically attracted for long enough time. While individual trajectories are chaotic, the chaotic attractor reveals information about the long-term trends of the system. The stretching causes orbits on the attractor to exhibit sensitive dependence on initial conditions (chaos) and the folding causes the fractal (strange) structure. The impressive structure appearing for $\gamma = 1.38$ is chaotic and is represented with the associated time series of the coordinates in *Figure 7*. The time series change from the quasi periodic shape that we have already encountered in previous simulations to a totally "random" shape. The equilibrium is $c_* = 0.2213, l_* = 1.3280$ and the eigenvalues are $\lambda_{1,2} = 0.1000 \pm 1.2449i$ with $|\lambda_{1,2}| \cong 1.5599$. The strange attractor is produced by the breaking of the invariant circles and the appearance of the seventeen chaotic regions changes as they are linked into a single chaotic attractor. Full developed chaos occurs also for $\gamma = 1.68$, after passing the second window of multiple Neimark-Sacker bifurcations for the iterates of the original map (see *Figure 8*). The fractal structure of the attractors is evident in both cases. To confirm this we compute the correlation dimension and obtain that $D_C = 1.0483$ for the
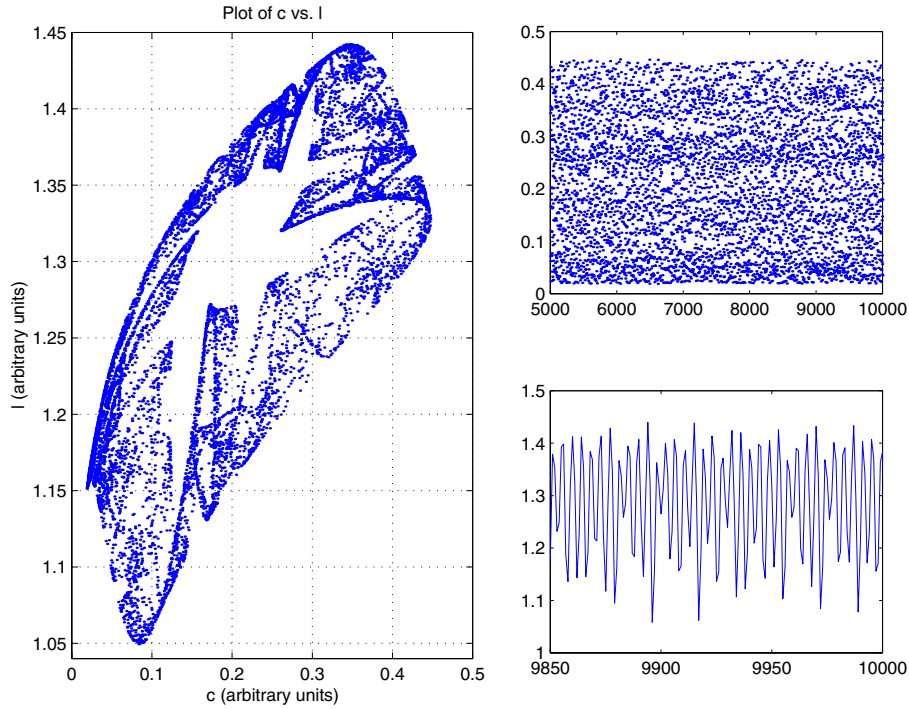
Figure 7. $\gamma = 1.38$, chaotic attractor.

attractor in *Figure 7* ($\gamma = 1.38$) and $D_C = 1.6243$ for the attractor in *Figure 8* ($\gamma = 1.68$). The estimation of the correlation dimensions is shown in *Figure 9*.

The Lyapunov exponents represent a dynamic measure of chaos that average the separation of the orbits of nearby initial conditions as the system moves forward in time. Each one of the chaotic attractors displayed in *Figure 7* and *8* has a positive Lyapunov exponent estimated to be $\lambda_1 = 0.0519$ for $\gamma = 1.38$ (the other Lyapunov exponent is negative and is equal to $\lambda_2 = -1.0779$) and $\lambda_1 = 0.2030$ for $\gamma = 1.68$ (with $\lambda_2 = -0.3225$). The convergence of the Lyapunov numbers (e^{λ_1}) is ilustrated in *Figure 10*. We use around 15.000 iterates in order to estimate the Lyapunov exponents.

Once known the Lyapunov exponents, we can obtain the information dimension using the Kaplan-Yorke Conjecture, that is

$$D_I \;\; = \;\; 1 + \frac{\lambda_1}{|\lambda_2|} = 1.0481, \text{ for } \gamma = 1.38 \text{ and}$$

$$D_I \;\; = \;\; 1 + \frac{\lambda_1}{|\lambda_2|} = 1.6293, \text{ for } \gamma = 1.68.$$

It is easy to observe that in both cases the correlation dimension and the information dimension are equal for the strange attractors presented in Figure 7 and 8. This shows the very complex nature of the dynamics produced by the OLG economic model.

4. Controlling Chaotic Economic Motion

In principle the control of chaotic systems does not differ from the control of general nonlinear systems. However, there is a substantial difference which was elegantly summarized by Mees
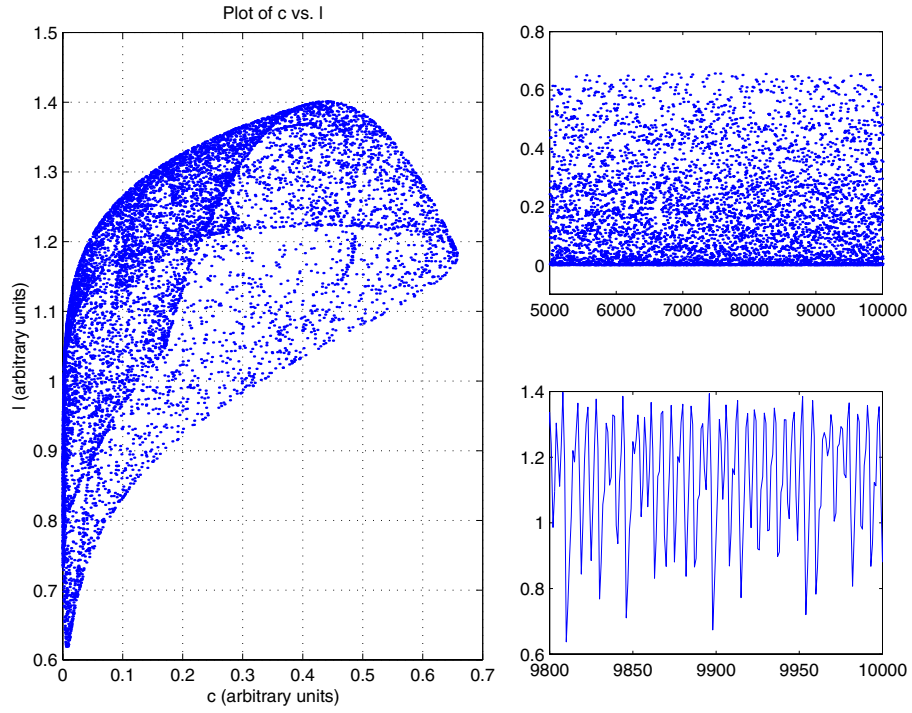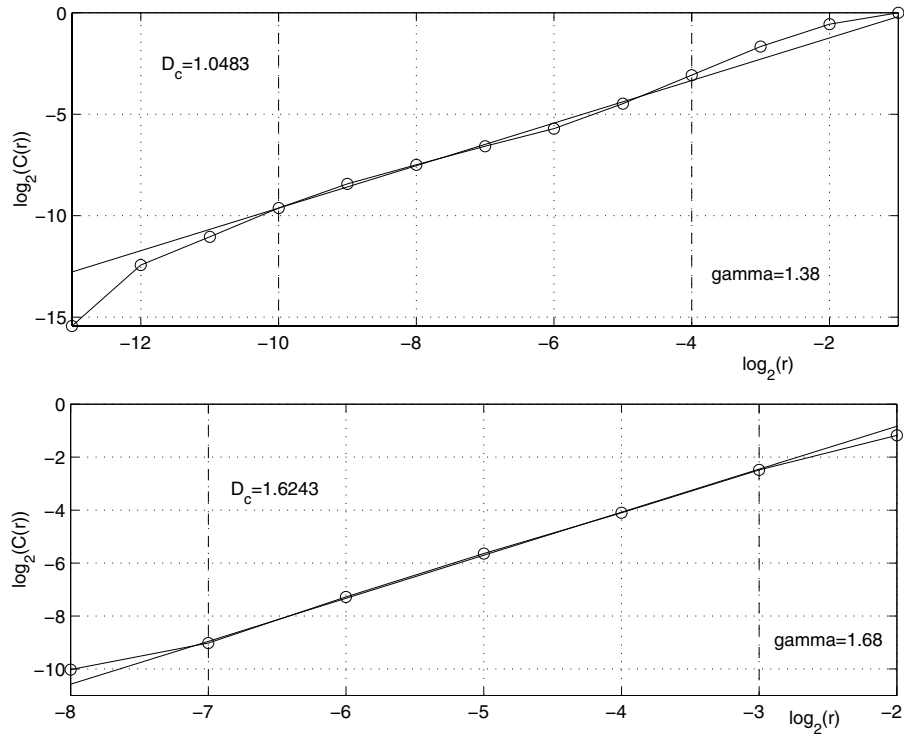
Figure 8. $\gamma = 1.68$, full developed chaos.

Figure 9. Correlation dimension for the two strange attractors obtained for $\gamma = 1.38$ and $\gamma = 1.68$.

Convergence of the maximal Lyapunov number for gamma = 1.38

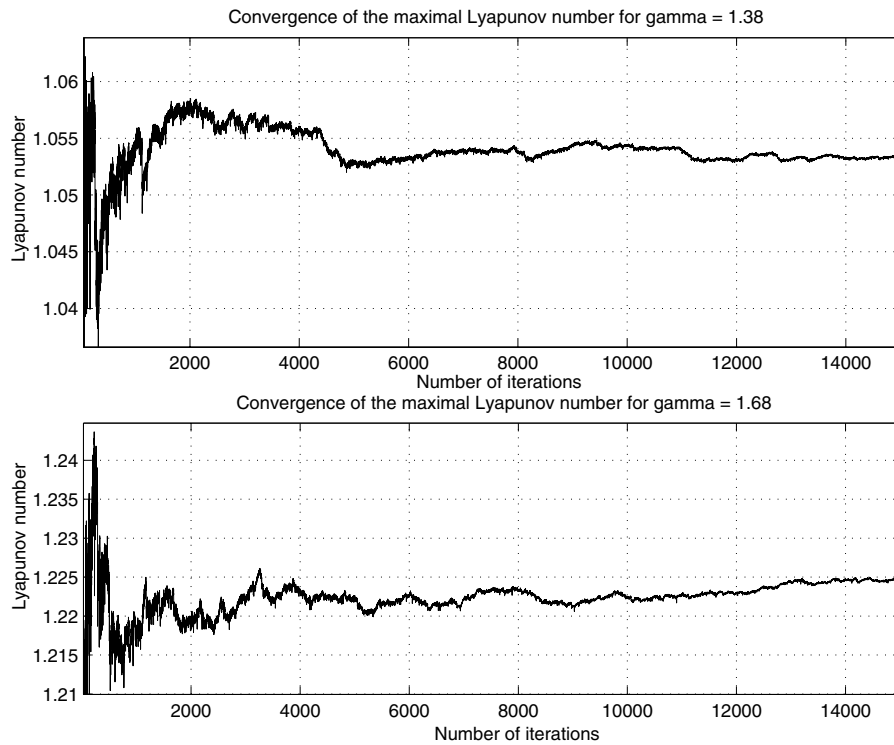Convergence of the maximal Lyapunov number for gamma = 1.68

Figure 10. Positive Lyapunov exponents corresponding to the strange attractors for $\gamma = 1.38$ and $\gamma = 1.68$.

(1998): "if we ask the right sort of questions, questions that may differ from those normally asked by a control theorist, we may be able to get a chaotic system to do something desirable *with rather little control effort on our part*. I call this *control by smart butterflies*, because of the infamous butterfly effect, which says that chaotic systems are sensitive to small changes" (1998, vii). That is, conventional classical control techniques control the dynamics of nonlinear processes through the use of brute force, having in fact frequently to change the nature of the very system that is subject to control because these systems are not sensitive to small changes in their parameters. However, in the case of chaotic systems, as these are sensitive to very small changes in the parameters, a small butterfly effect in one of them is (in most cases) all that is required to control their outcome, without changing the very nature of the controlled system in any relevant way. In short, if a chaotic system as an unstable fixed point, the control procedure turns this unstable point into a stable one, by a very small perturbation, leaving the system's initial fundamental characteristics untouched.

In general, the techniques for feedback control of chaos presented thus far in the literature have some common features which we will briefly summarize. The control is usually designed for parameter values where the system is known to exhibit chaotic motion, and is typically of the form $u = u(x - x_*)$ where x is the system state vector, and x_* is an unstable equilibrium of interest, which lies on a chaotic attractor. The control function u is not necessarily smooth. Thus, when an input is altered on the basis of the difference between the actual output of the system and the desired output, the system is said to involve feedback. Note that x_* can also be a periodic orbit. The Ott-Yorke-Grebogi method (OGY method) (1990) and the pole-placement technique (see Ogata (1997) and Romeiras *et al.* (1992)) belong to feedback control. The pole-

placement method extends that of OGY, allowing for a more general choice of the feedback matrix and implementation to higher-dimensional systems.

In what follows we will apply the pole-placement method to the CRRAL economic model that we have been discussing along this paper and we stabilize a unstable period-one orbit embedded in the chaotic attractor. By applying small, adequate chosen temporal perturbations to an accessible control parameter of the dynamical system, the original chaotic trajectory can be converted into the desired stable fixed point. The control parameter that we will use is the disutility of labor relative risk aversion coefficient, γ.

4.1. Controlling through γ by pole-placement technique

It was showed numerically in the previous section that for $\gamma = 1.38$, $b = 1.2$, $\alpha = \theta = 0.2$ the system exhibits a chaotic attractor (*Figure 7*). We fix these parameter values and consider that γ is the control parameter which is available for external adjustment but is restricted to lie in some small interval $|\gamma - \gamma_0| < \delta, \delta > 0$ around the nominal value $\gamma_0 = 1.38$. The system becomes:

$$\begin{cases} f : c_{t+1} = \left(l_t^\gamma - c_t^{0.2}\right)^{1/0.2} \\ \\ g : l_{t+1} = 1.2 \left(l_t - c_t\right) \end{cases} \tag{17}$$

We vary the control parameter γ with time t in such a way that for almost all initial conditions the dynamics of the system converge to the desired period one orbit contained in the attractor. The control strategy is the following: we find a stabilizing local feedback control law which is defined in a neighborhood of the desired periodic orbit. This is done by considering the first order approximation of the system at the chosen unstable periodic orbit. The ergodic nature of the chaotic dynamics ensures that the state trajectory eventually enters into the neighborhood. Once inside the neighborhood, we apply the stabilizing feedback control law in order to steer the trajectory towards the desired orbit.

In this case we consider the stabilization of the unstable period-one orbit $E_2 : (c_*, l_*) = (0.2213, 1.3280)$. The map can be approximated in the neighborhood of the fixed point by the following linear map,

$$\begin{bmatrix} c_{t+1} - c_* \\ l_{t+1} - l_* \end{bmatrix} \cong A \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} + B[\gamma - \gamma_0] \tag{18}$$

where

$$A_{(2\times2)} = \begin{bmatrix} \dfrac{\partial f\left(c_*, l_*\right)}{\partial c_t} & \dfrac{\partial f\left(c_*, l_*\right)}{\partial l_t} \\ \\ \dfrac{\partial g\left(c_*, l_*\right)}{\partial c_t} & \dfrac{\partial g\left(c_*, l_*\right)}{\partial l_t} \end{bmatrix} \quad \text{and} \tag{19}$$

$$B_{(2\times1)} = \begin{bmatrix} \dfrac{\partial f\left(c_*, l_*\right)}{\partial \gamma} \\ \\ \dfrac{\partial g\left(c_*, l_*\right)}{\partial \gamma} \end{bmatrix} \tag{20}$$

are the Jacobian matrixes with respect to the control state coordinates (c_t, l_t) and to the control parameter γ. The partial derivatives are evaluated at the nominal value γ_0 and at (c_*, l_*). In our case we get

$$\begin{bmatrix} c_{t+1} - 0.22 \\ l_{t+1} - 1.32 \end{bmatrix} \cong \begin{bmatrix} -1.0 & 2.29 \\ -1.2 & 1.2 \end{bmatrix} \begin{bmatrix} c_t - 0.22 \\ l_t - 1.32 \end{bmatrix} + \begin{bmatrix} 0.62 \\ 0 \end{bmatrix} [\gamma - 1.38] \tag{21}$$

Next, we check whether the system is controllable. A controllable system is one for which a matrix $H_{(1 \times n)}$ can be found such that $A - BH$ has any desired eigenvalues. This is possible if $\operatorname{rank}(C) = n$, where n is the dimension of the state space, and

$$C = [B : AB : A^2 B : ... : A^{n-1} B]. \tag{22}$$

In our case it follows that

$$C = [B : AB] = \begin{bmatrix} 0.62 & -0.62 \\ 0 & -0.75 \end{bmatrix} \tag{23}$$

which obviously has rank 2, and so we are dealing with a controllable system.

If we assume a linear feedback rule (control) for the parameter γ of the form

$$[\gamma - \gamma_0] = -H \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \tag{24}$$

where $H_{(1 \times 2)} := [h_1 \quad h_2]$, then the linearized map becomes

$$\begin{bmatrix} c_{t+1} - c_* \\ l_{t+1} - l_* \end{bmatrix} \cong [A - BH] \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \tag{25}$$

that is

$$\begin{bmatrix} c_{t+1} - 0.22 \\ l_{t+1} - 1.32 \end{bmatrix} \cong \begin{bmatrix} -1.0 - 0.62 h_1 & 2.29 - 0.62 h_2 \\ -1.2 & 1.2 \end{bmatrix} \begin{bmatrix} c_t - 0.22 \\ l_t - 1.32 \end{bmatrix} \tag{26}$$

which shows that the fixed point will be stable provided that the (2×2)-matrix $A - BH$ is asymptotically stable, that is, all its eigenvalues have modulus smaller than one. The eigenvalues μ_1, μ_2 of the matrix $A - BH$ are called the "regulator poles" and the problem of placing these poles at the desired location by choosing H with A, B given is the "pole-placement problem". If the controllability matrix C from equation (22) is of rank n, $n = 2$ in our case, then the pole-placement problem has a unique solution. This solution is given by

$$H = [\alpha_2 - a_2 \quad \alpha_1 - a_1] T^{-1} \tag{27}$$

where $T = CW$ and

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -0.20 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $\{a_1, a_2\}$ are the coefficients of the characteristic polynomial of A, i.e.,

$$\begin{aligned} |A - \lambda I| &= \lambda^2 + a_1 \lambda + a_2 = \lambda^2 - 0.20\lambda + 1.559 \\ &\Rightarrow \quad a_1 = -(\lambda_1 + \lambda_2) = -0.20, \; a_2 = \lambda_1 \lambda_2 = 1.559 \end{aligned}$$

and $\{\alpha_1, \alpha_2\}$ are the coefficients of the desired characteristic polynomial of $A - BH$, i.e.,

$$
\begin{aligned}
|(A - BH) - \mu I| &= \mu^2 + \alpha_1 \mu + \alpha_2 \\
&\Rightarrow \alpha_1 = -(\mu_1 + \mu_2) \\
&\Rightarrow \alpha_2 = \mu_1 \mu_2
\end{aligned}
$$

From equation (27) we get that

$$
\begin{aligned}
H &= \begin{bmatrix} \mu_1 \mu_2 - 1.559 & -(\mu_1 + \mu_2) + 0.20 \end{bmatrix} \begin{bmatrix} 0 & -1.32 \\ 1.59 & -1.59 \end{bmatrix} \\
&= \begin{bmatrix} -1.59(\mu_1 + \mu_2) + 0.318 & -1.32\mu_1\mu_2 + 1.59(\mu_1 + \mu_2) + 1.752 \end{bmatrix}.
\end{aligned}
$$

Since the CRRAL map is nonlinear, the application of linear control theory will succeed only in a sufficiently small neighborhood U around (c_*, l_*). Taking into account the maximal allowed deviation from the nominal control parameter γ_0 and equation (24), we obtain that we are restricted to the following domain

$$
S_H = \left\{ (c_t, l_t) \in \mathbb{R}^2 : \left| H \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \right| \le \delta \right\}. \tag{28}
$$

This defines a slab of width $2\delta / |H|$ and thus we activate the control (24) only for values of (c_t, l_t) inside this slab, and choose to leave the control parameter at its nominal value when (c_t, l_t) is outside the slab.

Any choice of regulator poles inside the unit circle serves our purpose. There are many possible choices of the matrix H. In particular, it is very reasonable to choose all the desired eigenvalues to be equal to zero and in this way the target would be reached at least after n periods, and, therefore, a stable periodic orbit is obtained out of the chaotic evolution of the dynamics.

The time efficiency in the control process is another issue that can also be considered. Romeiras *et al.* (1992) investigated numerically which choice of the feedback matrix H exhibits the shortest stabilization time. Because the linearized equation (25) does not take into account any nonlinearity that is part of the original chaotic system, the control may not be able to bring the orbit to the fixed point, despite the fact that it is already in the slab. In this case, the orbit will leave the slab and continue to wander chaotically as if there were no control. Since the orbit on the uncontrolled chaotic attractor is ergodic, at some time it will eventually satisfy the condition (28) and also be sufficiently close to the desired fixed point so that control is achieved. Thus, a stable orbit is created, which for a typical random initial condition, is preceded by a chaotic transient in which the orbit is similar to other orbits on the uncontrolled chaotic attractor. The length τ of such chaotic transient depends sensitively on the initial conditions of the particular orbit. For initial conditions randomly chosen in the basin of attraction the distribution of chaotic transient lengths is exponential

$$
\phi(\tau) = \frac{1}{\langle \tau \rangle} \exp\left(-\frac{\tau}{\langle \tau \rangle} \right)
$$

for large τ. The quantity $\langle \tau \rangle$ is the characteristic length of chaotic transient, called in the present case the average time to achieve control.

The transient phase where the orbit wanders chaotically before control being applied can be shortened by applying the targeting technique proposed by Shinbrot *et al.* (1990). It was
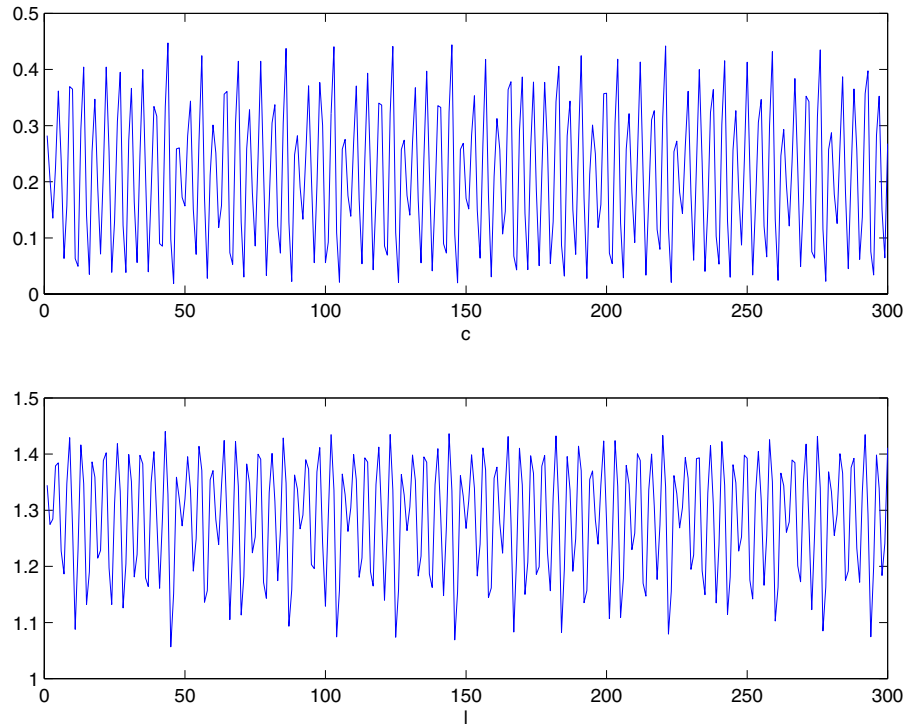
Figure 11. Original chaotic orbit.

pointed out that orbits can be rapidly brought to a target region on the attractor by using small control perturbations when the orbit is far from the neighborhood of the periodic orbit to be stabilized. The idea is that, since chaotic systems are exponentially sensitive to perturbations, these perturbations produce after some time a large effect on the orbit location and can be used to guide it.

4.2. Numerical examples
Two cases will be illustrated. Firstly, we consider different values of h_1 and h_2 and fix a certain orbit initiated at a particular point in the basin of attraction. As we will see, the controlled orbit will converge towards the fixed point but takes different periods of time in order to fully accomplish that convergence process, depending on the values of h_1 and h_2. Secondly, the chaotic trajectory will also converge to the fixed point if, in contrast, we consider fixed values of h_1 and h_2 and *randomly* choose some initial conditions. In all examples we iterate the system for 100 iterations until the chaotic behavior is perfectly evident and the iterates are distributed over the attractor and then apply the control strategy when the orbit is inside the slab. After this time point the system is forced to follow the desired orbit.

In *Figure 11* we show the time series of the chaotic trajectory initiated at the point $(c_0, l_0) = (0.320, 1.427)$ which we have chosen to control. In contrast, *Figure 12* presents the controlled orbit converging to the stabilized fixed point when the feedback matrix H is chosen such that the eigenvalues of $(A - BH)$ are $\mu_1 = \mu_2 = 0$. This implies that $\mu_1 + \mu_2 = 0$, $\mu_1 \mu_2 = 0$ and so $H = [0.318 \ 1.752]$. For this control strategy we have also chosen $\delta = 0.1$.

For $H = [-1.28 \ 3.01]$ the motion will converge to the stable orbit of period one which is showed in *Figure 13*. The matrix $A - BH$ has a pair of real eigenvalues equal to $\mu_1 = \mu_2 = 1.2$. The linear control is activated for theses values and for the time index 100. After switching on
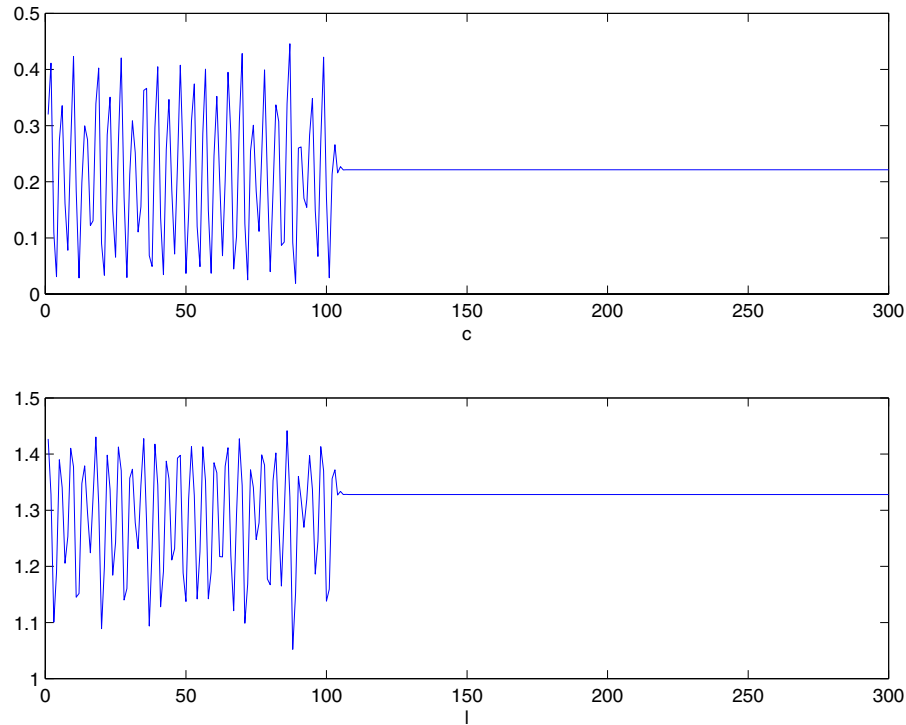
Figure 12. Controlled chaotic orbit for $H = [0.3184 \quad 1.7528]$.

the control, the orbit continues to exhibit chaotic behavior for some time, unchanged from the uncontrolled case, because it is not closed enough to the fixed point. After some steps, this is eliminated and the orbit is rapidly brought to the fixed point. We can observe that in this case, the orbit to enter the slab and the control to be achieved both will take a longer time span to be accomplished in comparison to the first example.

In what follows, we will place the poles such that $H = [0.318 \quad 1.752]$ and will consider randomly chosen initial conditions. *Figures 14, 15* show clearly that the chaotic transient depends sensitively on the initial conditions of the particular orbits.

For $\delta = 0.1$ and randomly chosen initial conditions, the pole placement control strategy works very well for this system. Exhaustive numerical experimentations show that almost all initial conditions lead to controllable orbits and the time to achieve the control is no longer than 100 iterates. For $\gamma = 1.68$, $b = 1.2$, $\alpha = \theta = 0.2$ analogous control results were obtained.

Further numerical experimentations were done. For an external adjustment such that the interval in which the control parameter can vary is smaller than in the previous examples, e.g., for $\delta = 0.01$, and for randomly chosen initial conditions, the control strategy still works very well. However, as one would expect the time span needed to stabilize the randomly chosen trajectory is much larger because of the smaller value for δ. In *Figure 16* we show that it takes around 150 iterates to have a randomly chosen trajectory under control for this case. However, for other initial conditions, the time span required to achieve control becomes much larger.

5. Summary and Conclusion

We have applied the pole-placement control technique to an overlapping generations model with production and an endogenous intertemporal decision between labour and leisure. It was showed
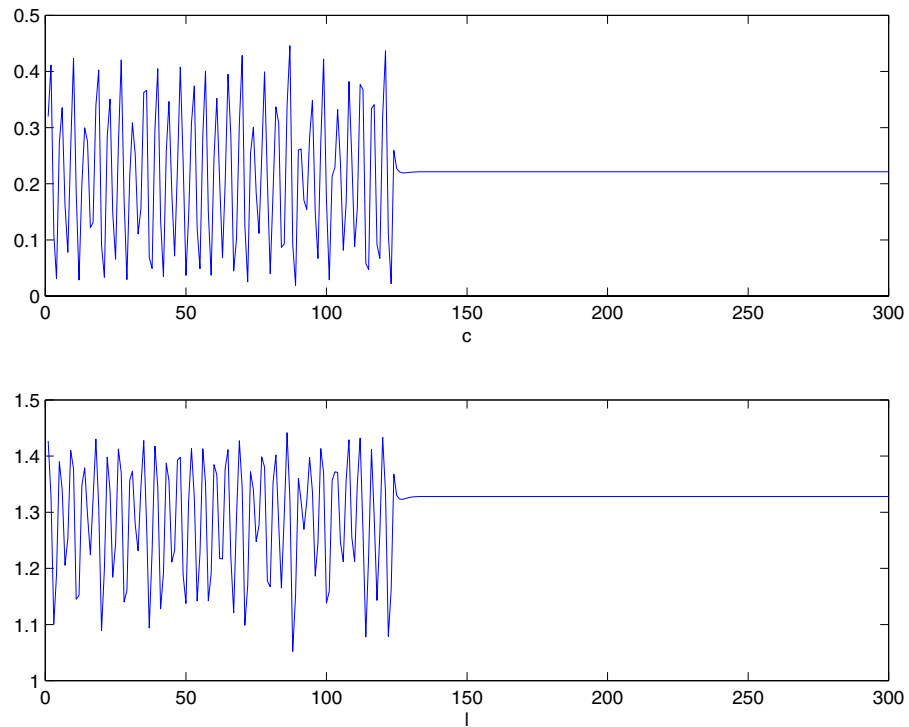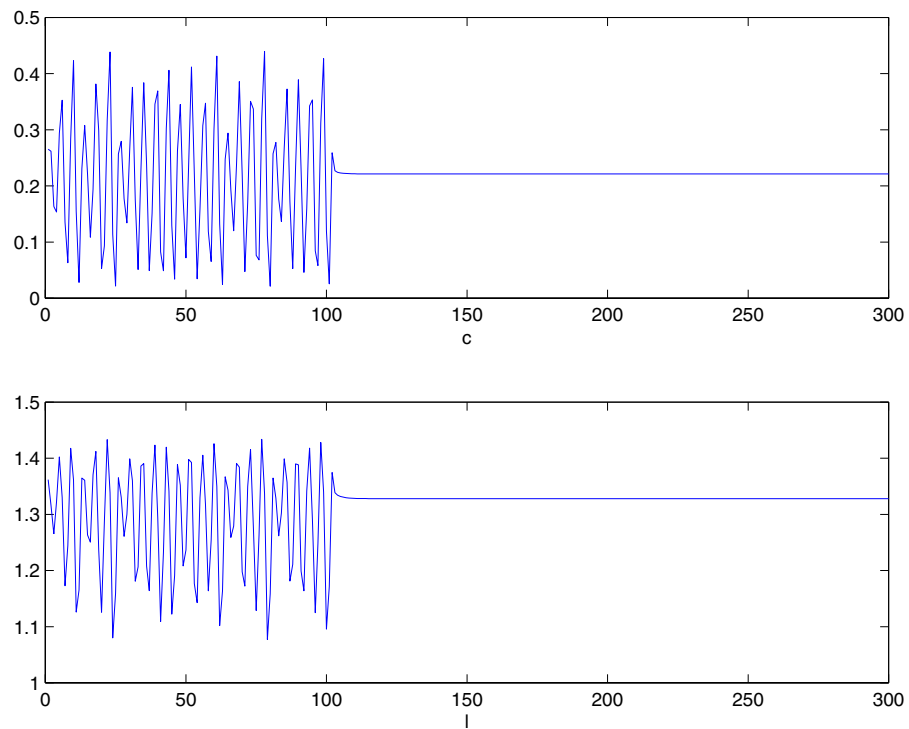
Figure 13. Controlled chaotic orbit for $H = [-1.28 \quad 3.01]$.

Figure 14. Controlled chaotic orbit for $H = [0.3184 \quad 1.7528]$ and randomly chosen initial values.
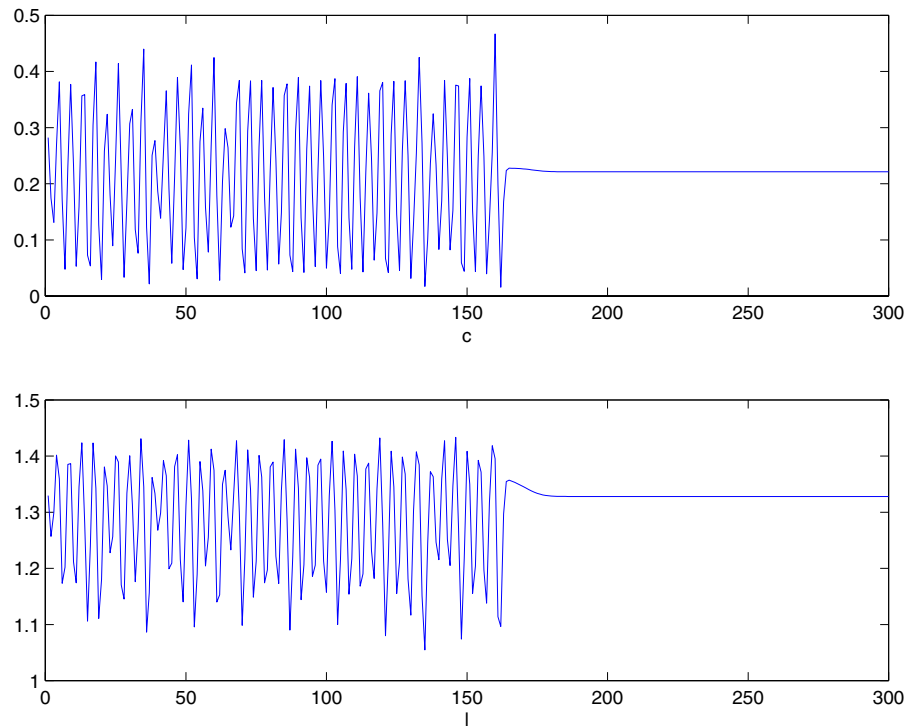
Figure 15. Controlled chaotic orbit for $H = [0.3184 \quad 1.7528]$ and randomly chosen initial values.
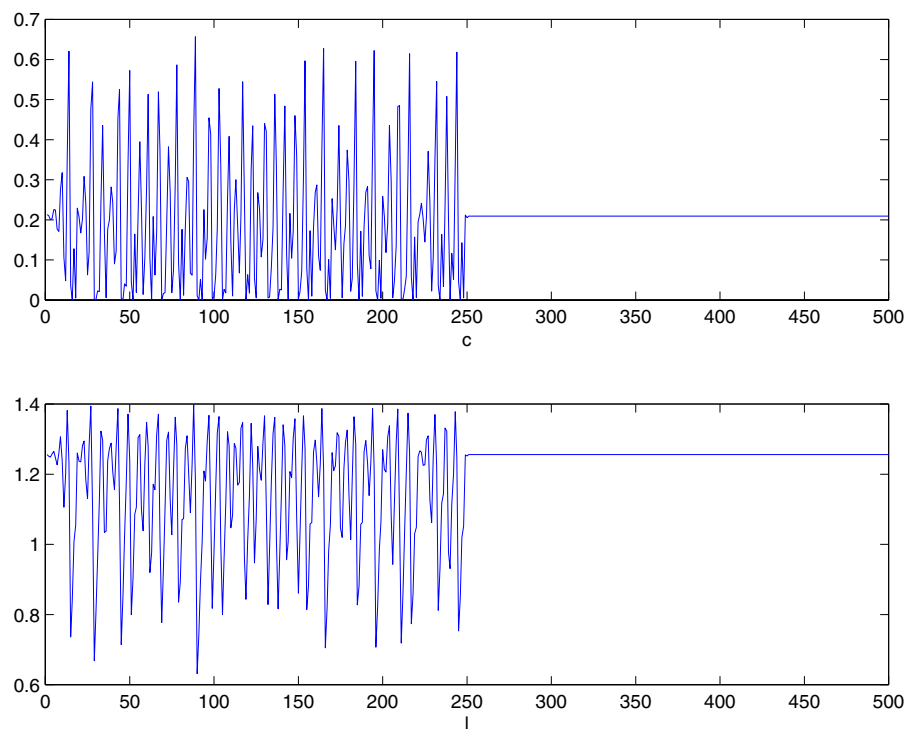

Figure 16. Controlled chaotic orbit for $\delta = 0.01$.

that the aperiodic and complicated motion that arises from the dynamics of the model can be easily subject to control by small perturbations in its parameters and be turned into a stable steady state. This simple exercise may raise serious reservations to the recent views sustaining that economic policy becomes impossible or useless in the presence of chaotic motion, at least on purely logical and mathematical grounds.

Two major points should be stressed. Firstly, the fine tuning of the system (that is, the control) can in fact be performed without having to rely *only* on infinitesimal accuracy in the perturbation process, because the control can be performed with larger or smaller perturbations, but neither too large (because these would change the chaotic initial fixed point, modifying therefore the nature of the system), nor too small because the control becomes too inefficient. In this paper we assumed that γ was the control parameter which was available for external adjustment and was restricted to lie in some small interval $|\gamma - \gamma_0| < \delta$, $\delta > 0$ around the nominal value $\gamma_0 = 1.38$. Two values for δ were considered: $\delta = 0.1$ and a much smaller interval $\delta = 0.01$. In both cases the control was easily achieved using either randomly or arbitrarily chosen initial conditions.

Secondly, the fundamental characteristics of the model are not changed by the control procedure as the fixed point that forms the basis of attraction remains the same — all that is changed is its stability properties: from unstable to a stable one — and the large cycles are eliminated. Therefore, instead of rendering economic policy useless, chaotic motion may in fact even vindicate the intervention of public agencies in models where such intervention would not be largely justified in accordance with conventional theory.

Despite the fact that on purely logical or mathematical terms it is perfectly possible to control chaotic motion, imposing larger or smaller external perturbations and accepting larger or smaller control time spans, when we turn to particular applications of control in specific fields of nature and society we must add a word of caution. In some areas it is extremely difficult (or simply, not possible) to control a dynamic system because external observers do not have power to influence the crucial parameters of the system. This is the case, for example, of weather control. Freeman Dyson was so critical of von Neumann because, apparently, the latter intended to use sophisticated models and the state of the art in computers to understand and control the weather system in the 1950s. We may be able to understand how the weather system behaves over time. But how can we directly influence (in the short term) the crucial parameters of such a gigantic process in order to control it? We are not aware of a practical way to be successful in such an endeavor.

However, this is not what happens in the fields economics and finance. In the case of economics, public agencies have many policy instruments that can be used to control the dynamics of the system. This should not be interpreted as suggesting that there is clear evidence of chaotic motion in economic and financial structures. In fact, there is a large controversy on this issue, with arguments in favour presented by Barnett and Chen (1988), De Grauwe and Grimaldi (2002), or DeCoster and Mitchell (1991) among others, while Serletis and Shintani (2004) or Ramsey and Rothman (1994) find strong rejection of chaos in economic time series. What seems highly doubtful is the relevance of the standard argument which sustains that the dynamics of aggregate economic variables (i.e., fluctuations of economic activity) can be properly described as the mere result of linear stochastic processes.

Therefore, if there is some form of relevant nonlinearity in the structure of modern economies (which we believe to be the case, see evidence in De Grauwe and Grimaldi), then the control of such structures may benefit a lot from the understanding of what chaos control is all about. For example, take the case of monetary policy in the US over the last few years. The FED have tried to control every signal of a serious unstable motion in the economy by small pushes and pulls applied at the right places on short term interest rates. Firstly, when the economy showed signs of a serious slowdown, the FED reduced interest rates by very small fractions (usually a

quarter of a percentage point), and more recently it increased short term interest rates also by small fractions when overheating became evident. This kind of small pushes and pulls at the right places, without changing the state of the economy apart from reducing or eliminating the fluctuations, seem to have some resemblance with the control of chaotic systems as we illustrated above, but not with the control of linear stochastic processes. In this kind of processes, policy can do nothing to prevent or reduce fluctuations, and if implemented changes the fixed point of the economy in a relevant way. This is not so if economic dynamics are somehow influenced by chaotic motion.

Acknowledgments

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