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COHOMOLOGY OF DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT

This article presents a detailed treatment of Livschitz theorem for hyperbolic diffeomorphisms. Based on the periodic data, this theorem provides a necessary and sufficient condition so that cohomological equations have sufficiently regular solutions. It is one of the main tools to obtain global data of cohomological nature from the periodic data. Since it is crucial in the statement of Livschitz theorem, it is also presented the Anosov closing lemma.

Key words: cocycles, cohomological equations, periodic data, Livschitz theorem, Anosov closing lemma

1. INTRODUCTION

In dynamical systems theory some issues of considerable importance can be reduced to solving an equation of the form

$$\varphi = \Phi \circ f - \Phi \quad (1.1)$$

where $f: X \to X$ is a dynamical system, $\varphi: X \to \mathbb{R}$ is a function, both known, and $\Phi: X \to \mathbb{R}$ is the unknown. The equation (1.1) is designated by cohomological equation. The study of cohomological equations relates particularly to the study of conjugacy to an irrational rotation of the circle, the existence of absolutely continuous measures for expansive circle maps and topological stability of hyperbolic torus automorphisms. These equations also arise naturally in statistical mechanics.

Some results established by Livschitz in the beginning of the 70s (1,2) specifically discuss the possibility of obtaining the cohomological equations solutions in the hyperbolic dynamics context. Given an hyperbolic dynamical system, the Livschitz theorem provides a necessary and sufficient condition, based only on the data given by the periodic orbits, to the existence of Hölder solutions.

After introducing key concepts such as the cocycle and cohomology between cocycles, and emphasizing the relationship between the existence of solutions of cohomological equations and the cocycles behaviour over periodic orbits, it is presented a detailed proof of the Livschitz theorem for hyperbolic diffeomorphisms. Previously it is outlined the proof of the Anosov closing lemma for diffeomorphisms since it provides a crucial estimate for the Livschitz theorem's proof.

2. COCYCLES AND COHOMOLOGY IN DISCRETE TIME

Let $T: \mathbb{Z} \times X \to X$ be a dynamical system with phase space X and discrete time. So, they are valid the group properties

$$T(m+n, x) = T\big(m, T(n, x)\big)$$

and $T(0, x) = x$, for each $x \in X$ and $n, m \in \mathbb{Z}$. Given $n \in \mathbb{Z}$, let $T(n): X \to X$ be the map defined by $T(n)x = T(n, x)$ through the dynamical system T. A *cocycle* over T is a function $\alpha: \mathbb{Z} \times X \to \mathbb{R}$ such that

$$\alpha(m+n, x) = \alpha(m, T(n)x) + \alpha(n, x) \quad (2.1)$$

whenever $x \in X$ and $n, m \in \mathbb{Z}$. Note that cocycles over T form a linear space. Defining, for each $n \in \mathbb{Z}$, the map $\tilde{T}(n): X \times \mathbb{R} \to X \times \mathbb{R}$ by $\tilde{T}(n)(x, y) = \big(T(n)x, y + \alpha(n, x)\big)$ then the property (2.1) is equivalent to $\tilde{T}(m+n) = \tilde{T}(m) \circ \tilde{T}(n)$.

Each function $\Phi: X \to \mathbb{R}$ induces a cocycle through the expression

$$\alpha(n,x) = \Phi(T(n)x) - \Phi(x) \qquad (2.2).$$

A cocycle defined in this way is called *coboundary*. A natural equivalence relationship between cocycles is the cohomology. Two cocycles α and β over a dynamical system T are called *cohomologous* if they differ by a coboundary, that is, if there is a function $\Phi : X \to \mathbb{R}$ such that

$$\alpha(n,x) - \beta(n,x) = \Phi(T(n)x) - \Phi(x).$$

Note that a cocycle α is a coboundary if and only if α is cohomologous to the trivial cocycle $\beta(n,x) = 0$; in this case α is called *cohomologically trivial* and any function Φ satisfying (2.2) is a *trivialization* of α. For a cocycle α to be a coboundary it is necessary that $\alpha(n,x) = 0$ for all values of $n \in \mathbb{Z}$ and $x \in X$ such that $T(n)x = x$. The equation (2.2) is called a *cohomological equation*.

3. DISCRETE PERIODIC DATA

Each cocycle α over a discrete dynamical system $T: \mathbb{Z} \times X \to X$ is uniquely determined by the function $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = \alpha(1,x)$. In fact, it is immediate to verify that

$$\alpha(n,x) = \begin{cases} \sum_{i=0}^{n-1} \varphi(T^i x) \text{ if } n > 0 \\ -\sum_{i=n}^{-1} \varphi(T^i x) \text{ if } n \leq 0 \end{cases} \qquad (3.1)$$

where $T^0 x = x$ and $Tx = T(1,x)$. So we can identify the dynamical system with the invertible map $T: X \to X$ (there being no notation confusion danger) which inverse is given by $T^{-1}x = T(-1, Tx)$. Accordingly there is a biunivocal correspondence between cocycles and real functions defined on X. Two functions $\varphi, \psi : X \to \mathbb{R}$ are called *cohomologous* (respecting to T) if

$$\varphi - \psi = \Phi \circ T - \Phi$$

for some function $\Phi : X \to \mathbb{R}$. We can easily verify that two functions are cohomologous if and only if the respective cocycles are cohomologous. Furthermore, a function is called a coboundary if it is cohomologous to the zero function.

Given a function $\varphi : X \to \mathbb{R}$, let α be a cocycle over T defined by (3.1). To show that the equation, also called cohomological equation,

$$\varphi = \Phi \circ T - \Phi \ (3.2)$$

has a solution it is equivalent to show that the cocycle α is a coboundary. In fact, if the cohomological equation (3.2) is satisfied then, for each $n > 0$,

$$\alpha(n,x) = \sum_{i=0}^{n-1} \varphi(T^i x) = \sum_{i=0}^{n-1} [\Phi(T^{i+1}x) - \Phi(T^i x)] = \Phi(T^n x) - \Phi(x)$$

(with similar identities for $n \leq 0$) and α is a coboundary. On the other hand, if α is a coboundary then there is a function $\Phi : X \to \mathbb{R}$ such that

$$\alpha(n,x) = \Phi(T^n x) - \Phi(x).$$

Making $n = 1$ we conclude that the cohomological equation (3.2) is satisfied by Φ.

Suppose now that the cohomological equation (3.2) has a solution. If x is a m-periodic point of the dynamical system T, that is $T^m x = x$, then

$$\sum_{i=0}^{m-1} \varphi(T^i x) = \alpha(m,x) = \Phi(T^m x) - \Phi(x) = 0.$$

Therefore, it is necessary that $\sum_{i=0}^{m-1} \varphi(T^i x) = 0$ for all m-periodic point x so that there is a solution Φ to the cohomological equation (3.2).

If it is not require any additional structure to a solution of the cohomological equation then there is no difficulty in showing their existence, since $\sum_{i=0}^{m-1} \varphi(T^i x) = 0$ for each m-periodic point x; in fact, we can pick up one point x from each orbit of T, arbitrarily choose $\Phi(x) \in \mathbb{R}$ and then set Φ over the points of each orbit by

$$\Phi(T^n x) = \Phi(x) + \sum_{i=0}^{n-1} \varphi(T^i x).$$

If, however, there is some additional structure in the phase space X which we intend to maintain, this procedure may be unsatisfactory. For example, in the case of irrational rotation of the circle, this construction necessarily starts from a collection of non-measurable points and so, in general, we obtain a non-measurable solution of the cohomological equation.

4. LIVSCHITZ THEOREM FOR DIFFEOMORPHISMS

Let M be a Riemannian manifold, with norm $\|.\|_x$ and inner product $\langle . \rangle_x$ in the tangent space $T_x M$ of each point $x \in M$. In what follows we will write only $\|.\|$ and $\langle . \rangle$ without reference to the point x.

Let $f : M \to M$ be a diffeomorphism and $\Lambda \subset M$ an f-invariant set (that is, $f\Lambda = \Lambda$). A map $f|_\Lambda$ is *topologically transitive* if there is $x_0 \in M$ whose orbit $\{f^n x_0 : n \in \mathbb{Z}\}$ is dense in Λ. If there is an open neighborhood V of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n V$$

then Λ is *locally maximal* to f. An f-invariant set $\Lambda \subset M$ is *hyperbolic* to f if:

i. the tangent space restricted to Λ can be written as a continuous direct sum of df-invariant bundles, that is, for each $x \in M$ there is a decomposition of the tangent space $T_x M = E^s(x) \oplus E^u(x)$ which varies continuously with x and verifies $d_x f E^s(x) = E^s(fx)$ and $d_x f E^u(x) = E^u(fx)$;
ii. there are constants $C > 0$ and $\tau \in (0,1)$ such that, for each $x \in \Lambda$ and $n \in \mathbb{N}$, we have $\|d_x f^n v\| \le C\tau^n \|v\|$ for $v \in E^s(x)$, and $\|d_x f^{-n} v\| \le C\tau^n \|v\|$ for $v \in E^u(x)$.

$E^s(x)$ and $E^u(x)$ are called the *stable* and *unstable subspaces* at the point x, respectively.

Given a function $\varphi : X \to \mathbb{R}$, consider the cohomological equation (1.1), $\varphi = \Phi \circ f - \Phi$. As it is noted in previous Section 3, if this equation has a solution then it is immediate that $\sum_{i=0}^{m-1} \varphi(f^i x) = 0$ whenever x is a m-periodic point of f, that is, $f^m x = x$. Furthermore, we have shown that this is a necessary and sufficient condition for the existence of cohomological equation (1.1) solutions.

However, the solutions may be discontinuous or even not measurable. Naturally arise the question of how to ensure the existence of continuous solutions or even solutions with some additional regularity. The Livschitz theorem, formulated below, responds to this question in the hyperbolic dynamics context.

Livschitz Theorem: Let M be a Riemannian manifold and $f : M \to M$ be a C^1 diffeomorphism. Let $\Lambda \subset M$ be a compact invariant hyperbolic set locally maximal with $f|_\Lambda$ topologically transitive. Suppose that $\varphi : X \to \mathbb{R}$ is an Hölder function such that $\sum_{i=0}^{m-1} \varphi(f^i x) = 0$ whenever $f^m x = x$. Then there exist an Hölder function $\Phi : X \to \mathbb{R}$ with at least the same Hölder exponent as φ, and unique up to an additive constant, such that $\varphi = \Phi \circ f - \Phi$.

Given the relationship exposed in Section 3 between solving cohomological equations and obtaining coboundaries, we can interpret this theorem as follows: for hyperbolic dynamics and Hölder functions, the periodic data are necessary and sufficient to identify Hölder coboundaries.

The proof of the theorem, which closely follows the the suggestions of Katok and Hasselblatt in (3) (although not pragmatically oriented to the study of cohomology in dynamical systems), consists of the following steps:

i. the function Φ is determined along a dense orbit (guaranteed by the existence of topological transitivity of f in Λ) by choosing an arbitrary value in one of the orbit points;
ii. the Hölder regularity of φ is then used to ensure the Hölder regularity of Φ while Φ is extended to the whole set Λ.

The Anosov closing lemma is crucial for the first part of the Livschitz theorem's proof. It ensures, for diffeomorphisms with hyperbolic sets, that in the neighborhood of orbits returning sufficiently close to themselves there are periodic points. Moreover, the lemma estimated quantitatively how the constructed periodic orbit differs from initial orbit: it states how the distance between corresponding points of the initial orbit and the constructed periodic orbit is "controlled".

5. PROOF OF THE LIVSCHITZ THEOREM FOR DIFFEOMORPHISMS

Anosov Closing Lemma: Let M be a Riemannian manifold and $\Lambda \subset M$ a compact hyperbolic set locally maximal for the C^1 diffeomorphism $f : M \to M$. Then for all $\alpha \in (0,1)$ sufficiently large there is an open neighborhood V of Λ and constants $C, \delta > 0$ such that if $x \in \Lambda$ satisfies $d(f^n x, x) < \delta$ then exists a point $y \in \Lambda$ such that $f^n y = y$ and, for $k = 0,1, \dots, n$,

$$d(f^k x, f^k y) \leq C\alpha^{min\{k,n-k\}} d(f^n x, x) \tag{5.1}$$

This lemma is proved in (4) for $M = \mathbb{R}^n$. With the exception of inequality (5.1) statement, are followed in this proof the suggestions of Katok and Hasselblatt in (3).

Proof of the Livschitz Theorem: Since $f|_\Lambda$ is topologically transitive there is a point $x_0 \in \Lambda$ with orbit dense in Λ. By choosing an arbitrary real value $\Phi(x_0)$ define $\Phi(f^m x_0) = \Phi(x_0) + \alpha(m, x_0)$, where

$$\alpha(n,x) = \begin{cases} \sum_{i=0}^{n-1} \varphi(f^i x) \text{ if } n > 0 \\ -\sum_{i=n}^{-1} \varphi(f^i x) \text{ if } n \leq 0 \end{cases}.$$

Let $m, n \in \mathbb{N}$ such that $d(f^n x_0, f^m x_0)$ is small enough in order to apply the Anosov closing lemma. Assuming that $m > n$, this lemma provides constants $C > 0$, $\alpha \in (0,1)$ and a point $y \in \Lambda$ such that $y = f^{m-n} y$ and

$$d(f^{n+i} x_0, f^i y) \leq C\alpha^{min\{i,m-n-i\}} d(f^n x_0, f^m x_0) \tag{5.2}$$

for $i = 0,1, \dots, m-n$. Taking into account the definition of Φ in the dense orbit of x_0, observe that

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| = \left|\sum_{i=0}^{n-1} \varphi(f^i x_0) - \sum_{i=0}^{m-1} \varphi(f^i x_0)\right| = \left|\sum_{i=0}^{m-n-1} \varphi(f^{n+i} x_0)\right|.$$

Given the hypothesis concerning periodic points, we have

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| = \left|\sum_{i=0}^{m-n-1} [\varphi(f^{n+i} x_0) - \varphi(f^i y)]\right| \leq \sum_{i=0}^{m-n-1} \left|\varphi(f^{n+i} x_0) - \varphi(f^i y)\right|.$$

Since φ is Hölder with exponent $\theta \in (0,1]$, there is a constant $K > 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| \leq K\, d(x_1, x_2)^\theta.$$

Then we obtain

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| \leq \sum_{i=0}^{m-n-1} K\, d(f^{n+i} x_0, f^i y)^\theta.$$

It follows from (5.2) that

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| \leq \sum_{i=0}^{m-n-1} K \left(C\alpha^{min\{i,m-n-i\}} d(f^n x_0, f^m x_0)\right)^\theta.$$

So

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| \leq 2KC^\theta\, d(f^n x_0, f^m x_0)^\theta \sum_{i=0}^{m-n-1} \alpha^{\theta i} < 2KC^\theta\, d(f^n x_0, f^m x_0)^\theta \frac{1}{1-\alpha^\theta}.$$

Finally we obtain

$$\left|\Phi(f^n x_0) - \Phi(f^m x_0) < \frac{2KC^\theta}{1-\alpha^\theta}\, d(f^n x_0, f^m x_0)^\theta\right| \tag{5.3}$$

Similarly we can show that this inequality is also valid for any $m, n \in \mathbb{Z}$. Since Φ is Hölder in the orbit of x_0 and this orbi tis dense in Λ, the function Φ is uniquely extendable to a continuous function in Λ, which we still denote by Φ. Immediately follows from (5.3) that the extension of Φ into Λ has at least the same Hölder exponent as φ. Since f is continuous and φ is Hölder continuous, then φ and $\Phi \circ f - \Phi$ are continuous functions in Λ which coincide on the dense orbit of x_0. Therefore, they coincide in the whole set Λ and Φ is a continuous solution of the cohomological equation. The uniqueness is guaranteed because the kind of choice of $\Phi(x_0)$ determines uniquely the function Φ.

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