LE MATEMATICHE
Vol. LXI (2006) - Fasc. II, pp. 371-383

ON SOME POSITIVE EMBEDDINGS OF $\mathbb{P}^d$

FRANCESCA INCENSI - HELENA SOARES

We prove that any two embeddings $\mathbb{P}^d \cong Y \hookrightarrow X_1$, $\mathbb{P}^d \cong Y \hookrightarrow X_2$, $d \geq 3$, in two n-folds projective varieties X_1, X_2 with normal bundle $N_{Y|X_1} \cong X_{Y|X_2} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$ are formally equivalent i.e. $X_{1/Y} \cong X_{2/Y}$. Moreover, we see that Y is $G2$ in both X_1 and X_2. As an immediate consequence of this result and if furthermore Y is $G3$ in both X_1 and X_2 then we deduce that the two embeddings are Zariski equivalent.

1. Introduction.

The aim of this work is to study embeddings $Y \hookrightarrow X$ of a d-dimensional projective space $Y = \mathbb{P}^d$ into a projective n-fold X of dimension $n \geq 2$ with normal bundle $N_{Y|X}$ of Y in X isomorphic to

$$(n-d)\mathcal{O}_{\mathbb{P}^d}(1) := \mathcal{O}_{\mathbb{P}^d}(1)^{\oplus n-1},$$

i.e. to the normal bundle of a d-dimensional linear subspace of $\mathbb{P}^n$.

If $d = 1$ then Y is called a *quasi-line* in X, and the geometry of projective manifolds carrying quasi-lines was studied in [2], see also

Entrato in redazione il 30 Novembre 2006.
2000 *Mathematics Subject Classification*. Primary 14E25, 14B20; Secondary 14B10.
Keywords and phrases. Positive embeddings, formal equivalence, unirational varieties.

[1], chapter 14. It turns out that there are many interesting examples of complex projective manifolds carrying quasi-lines (some of which are Fano manifolds, see loc.cit.).

Throughout this paper we shall deal with the case with the case $d \geq 2$. When studying this kind of embeddings, formal geometry turns out to be a fruitful tool. In fact, the formal completion $X_{/Y}$ of X along Y is a very suitable object to study the given embedding and is an analogue of the concept of a tubular neighborhood of a submanifold of a complex manifold.

The theory of formal functions was introduced by Zariski to prove his famous connectedness theorem [14]. Later on, Grothendieck extended this theory and developed the language of formal schemes which gave him important tools to prove the fundamental Grothendieck's existence theorem [8] and other related questions (see [6] and [7]). Formal geometry has proven its importance not only for its strong connection with connectedness theorems but also for its relation with the question of extending meromorphic functions. For instance, besides allowing to prove Zariski's connectivity theorem mentioned above, it gave a new interpretation of Fulton-Hansen's connectedness theorem, providing a proof to a significant improvement of it as well as to some other applications (see e.g. [3], or [1], chapter 11). Also, the extension problem for meromorphic functions can be solved if it can be solved for formal functions (see [1], chapter 10).

The paper is organized as follows. In Section 2 we recall the main definitions and results that will be needed. In Section 3 we give an example of an embedding of $Y = \mathbb{P}^d$, with $d \geq 2$, in a projective manifold X of dimension $n \geq 2d + 1$ such that $N_{Y|X} \cong (n - d)\mathcal{O}_{\mathbb{P}^d}(1)$, which is formally equivalent (but not Zariski equivalent) to the linear embedding $\mathbb{P}^d \hookrightarrow \mathbb{P}^n$. This example leads us to the general setting of Section 4 of whether any two embeddings $\mathbb{P}^d \cong Y \hookrightarrow X_i, i = 1, 2$, with normal bundles $N_{Y|X_i}, \cong (n - d)\mathcal{O}_{\mathbb{P}^d}(1)$, $d \geq 1$, are formally equivalent and Zariski equivalent. When $d = 1$, this is a well-known case, see [2], or also [1]. In fact in this case Y is called a quasi-line, and there are a lot of examples of quasi-lines on projective n-folds which are very far from being formally equivalent to a line in $\mathbb{P}^n$. As far as the case $d \geq 2$ the situation is a lot more rigid. In fact our main theorem treats the case $d \geq 3$ and is the following:

Theorem 1.1. *Let $\mathbb{P}^d \cong Y \hookrightarrow X_1, \mathbb{P}^d \cong Y \hookrightarrow X_2$, with $d \geq 3$, be two embeddings in two n-folds projective varieties X_1, X_2 with normal bundle $N_{Y|X_1} \cong N_{Y|X_2} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$. Then the two embeddings are formally equivalent i.e. $X_{1/Y} \cong X_{2/Y}$. Moreover, Y is $G2$ in both X_1 and X_2.*

As a corollary of this result we get the following:

Corollary 1.2. *Under the same hypothesis of Theorem 1.1, if furthermore we assume Y is $G3$ in both X_1 and X_2, then the two embeddings are Zariski equivalent. In particular, X_1 and X_2 are rational.*

Finally, the case $d = 2$ seems to be a more delicate one and we are able to prove that:

Theorem 1.3. *Let $\mathbb{P}^2 \cong Y \hookrightarrow X_1, \mathbb{P}^2 \cong Y \hookrightarrow X_2$ be two embeddings in two n-folds projective varieties X_1, X_2 with normal bundle $N_{Y|X_i} \cong (n-2)\mathcal{O}_{\mathbb{P}^2}(1), i = 1, 2$. Denote by $Y(j)_i$ the j-th infinitesimal neighborhood of Y in $X_i, i = 1, 2$. Then $Y(2)_1 \cong Y(2)_2$ (inducing identity on Y). Moreover, the embeddings $Y \subset X_1$ and $Y \subset X_2$ are formally equivalent if and only if $Y(3)_1 \cong Y(3)_2$ (inducing identity on Y).*

Acknowledgements.

We would like to thank Professor Lucian Bădescu who proposed this problem to us and supervised our work, giving us very valuable comments and suggestions. This project started during our stay in the P.R.A.G.MAT.I.C. 2006 held at the Department of Mathematics of the University of Catania. We would also like to thank this institution for their warm hospitality and for the excellent working atmosphere. The second named author also acknowledges the partial support of Fundação para a Ciência e Tecnologia under grant SFRH/BD/16589/2004.

2. Background material.

Let us start by recalling some of the basic concepts and results that we will need in the sequel. Let X be an irreducible projective variety over an algebraically closed field k and let Y be a closed subvariety of X corresponding to an ideal sheaf $\mathcal{I}$ of the structural sheaf $\mathcal{O}_X$ of X.

Consider the formal completion $X_{/Y}$ of X along Y, that is,

$$X_{/Y} := \operatorname{dir}\lim_{n\geq 0} Y(n) = \bigcup_{n=0}^{\infty} Y(n),$$

where $Y(n) := (Y, \mathcal{O}_X/\mathcal{I}^{n+1})$ is the infinitesimal neighborhood of order n of Y in X.

Definition 2.1. Let $Y_1 \overset{i_1}{\hookrightarrow} X_1$ and $Y_2 \overset{i_2}{\hookrightarrow} X_2$ be two closed embeddings of projective varieties Y_1 and Y_2 into projective varieties X_1 and X_2, respectively. Then i_1 and i_2 are said to be Zariski equivalent if there exist open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$ containing Y_1 and Y_2 respectively, and an isomorphism $\varphi : U_1 \overset{\sim}{\to} U_2$ such that $\varphi(Y_1) = Y_2$.

The two embeddings are said to be formally equivalent if there exists an isomorphism $X_{1/Y_1} \cong X_{2/Y_2}$ between the formal completions X_{1/Y_1} and X_{2/Y_2}.

As usual, denote by $K(X)$ the ring of rational functions of X and by $K\big(X_{/Y}\big)$ the ring of formal-rational functions of X along Y. Then there is a canonical morphism $X_{/Y} \to X$ which induces a natural ring homomorphism

$$\alpha_{X,Y} : K(X) \longrightarrow K\big(X_{/Y}\big).$$

Definition 2.2. (Hironaka-Matsumura, [13]). Let X be a projective irreducible variety over an algebraically closed field k and let Y be a closed subvariety. We say that Y is $G3$ in X if the canonical map $\alpha_{X,Y} : K(X) \to K\big(X_{/Y}\big)$ is an isomorphism of rings. We say that Y is $G2$ in X if $K\big(X_{/Y}\big)$ is a field and the map $\alpha_{X,Y}$ makes $K\big(X_{/Y}\big)$ a finite field extension of $K(X)$.

In particular, if Y is $G3$ in X then $K\big(X_{/Y}\big)$ is a field. Moreover, being $G3$ means that every formal-rational function of X along Y can be extended to a usual rational function on X. Obviously, $G3$ condition implies $G2$ condition.

Theorem 2.3. (Hironaka-Matsumura, [13]). *Every positive-dimensional connected subvariety on the n-dimensional projective space $\mathbb{P}^n$ ($n \geq 2$) over an algebraically closed field k is $G3$ in $\mathbb{P}^n$.*

Proposition 2.4. (Gieseker, [5]). *Let $f : X' \to X$ be a finite morphism* *of algebraic varieties. Let Y be a closed subvariety of X such that the inclusion $i : Y \hookrightarrow X$ lifts to an embedding $i' : Y \hookrightarrow X'$ such that $f \circ i' = i$. Then f is étale in a Zariski open neighborhood of Y in X' if and only if the morphism of formal schemes $\widetilde{f} : X'_{/Y} \to X_{/Y}$ is an isomorphism.*

The next result, due to Hartshorne [12], is very important in what follows.

Theorem 2.5. (Hartshorne). *Let X be an irreducible projective variety of dimension ≥ 2 and let Y be a closed connected subvariety of X of dimension ≥ 1. Suppose $Y \subset \mathrm{Reg}(X)$ (i.e. Y is contained in the smooth locus of X) and Y is a local complete intersection in X. Then, if the normal bundle $N_{Y|X}$ of Y in X is ample then Y is $G2$ in X.*

Remark 2.6. Under the hypothesis of Theorem 2.5, Y need not to be $G3$ in X in general. We will see this fact in an example below. However, Harshorne conjectured in [10] that if Y is smooth, $N_{Y|X}$ is ample and $\dim(Y) \geq \frac{1}{2}\dim(X)$ then Y should be $G3$ in X. The case when X is a homogeneous space is already proved (see [1], chapter 13).

Theorem 2.7. (Gieseker). *Let (X_1, Y_1) and (X_2, Y_2) be two pairs each consisting of an irreducible projective variety X_i and a closed subvariety Y_i, $i = 1, 2$. If Y_i is $G3$ in X_i, $i = 1, 2$, and (X_1, Y_1) and (X_2, Y_2) are formally equivalent then (X_1, Y_1) and (X_2, Y_2) are also Zariski equivalent.*

One of the key tools that we will use in the next pages is the following result (see [9], Proposition (4.5)), whose proof makes essential use of Grothendieck's obstruction theory (see [7], éxp. III):

Theorem 2.8. (Grothendieck). *Let Y be a smooth projective variety of dimension $d \geq 1$, and let N be a vector bundle on Y such that the following condition holds:*

$$(1) \quad H^1(Y, T_Y \otimes S^m(N^*)) = H^1(Y, N \otimes S^{m+1}(N^*)) = 0, \quad \forall m \geq 1,$$

where $S(N^*) := \bigoplus_{m=0}^{\infty} S^m(N^*)$ is the symmetric $\mathcal{O}_Y$-algebra of the dual N^* of N. Then any nonsingular formal scheme $\mathfrak{X}$ with reduced scheme of definition Y and with normal bundle N, is isomorphic to the formal *completion of the geometric vector bundle* $\mathbb{V}(N^*) := \mathrm{Spec}(S(N^*))$ *along its zero-section.*

A consequence of Theorem 2.8 is the following:

Corollary 2.9. *Let* (Y, N) *be a pair as in Theorem 2.8 satisfying the cohomological conditions* (1), *and let* $Y \hookrightarrow X_i$, $i = 1, 2$ *be two closed embeddings, with* X_1 *and* X_2 *smooth varieties, such that the normal bundle of* Y *in* X_i *is isomorphic to* N, $i = 1, 2$. *Then the two embeddings are formally equivalent.*

3. An example.

Let us now give an example in order to better understand the problem we want to study. Let $X' := \mathbb{P}^n$ be the n-dimensional projective space over $\mathbb{C}$ and let $G :=< \xi >$ be the cyclic group, where ξ is a $n+1$ primitive root of the unity. Consider the action of G over X' given by

$$\begin{array}{ccc} G \times X' & \longrightarrow & X' \\ (\xi, [x_0 : \ldots : x_n]) & \mapsto & [x_0 : \xi x_1 : \xi^2 x_2 : \ldots : \xi^n x_n]. \end{array}$$

Let U' be the open subset of X' where G acts freely. Consider the d-linear subspace $L \cong \mathbb{P}^d \subset X'$, with $1 \le d \le \dfrac{n-1}{2}$ defined by equations

$$L : \begin{cases} x_0 = x_1 \\ x_2 = x_3 \\ \ldots \\ x_{2d} = x_{2d+1} \\ x_{2d+2} = \ldots = x_n = 0. \end{cases}$$

It is clear that if we consider the $n+1$ d-linear subspaces $L, \xi L, \ldots, \xi^n L$, these are pairwise disjoint and $\xi^i L \subset U'$ for all $0 \le i \le n$.

Let $\pi : X' = \mathbb{P}^n \to X := \mathbb{P}^n/G$ be the canonical morphism onto the quotient X and set $U := \pi(U')$ and $Y := \pi(L)$. Thus the restriction $\pi_{|U'}$ is an étale morphism of degree $n+1$, U is smooth and $\pi_{|L} : L \cong \mathbb{P}^d \xrightarrow{\sim} Y$.

$$\begin{tikzcd} \mathbb{P}^d \cong L \arrow[r, hook] \arrow[d, "\pi_{|L}", "\wr"'] & U' \arrow[r, hook] \arrow[d, "\pi_{|U'}"] & X' \cong \mathbb{P}^n \arrow[d, "\pi"] \\ Y \arrow[r, hook, "i"] & U \arrow[r, hook] & X \cong \mathbb{P}^n/G\,. \end{tikzcd}$$

Moreover π is an étale neighborhood of Y in X', i.e. the inclusion $i : Y \hookrightarrow U$ lifts to an inclusion $i' : Y \hookrightarrow X'$ such that $\pi \circ i' = i$ and π is étale at every point of $i'(Y)$. Hence, by Proposition 2.4, the formal completions $X'_{/Y}$ and $X_{/Y}$ are isomorphic, so the pairs (X, Y) and (X', Y) are formally equivalent. Nevertheless they are *not* Zariski equivalent since every open subset $V \subset U$ containing Y is not simply connected.

This way we may consider the following diagram

$$\begin{tikzcd} K(X) \arrow[r, hook, "\alpha_{X,Y}"] \arrow[d, "\pi^*"] & K(X_{/Y}) \arrow[d, "\widehat{\pi}^*", "\wr"'] \\ K(X') \arrow[r, "\sim", "\alpha_{X',Y}"'] & K(X'_{/Y}) \end{tikzcd}$$

By Theorem 2.3 of Hironaka-Matsumura, Y is $G3$ in X' and therefore $K(X') \cong K(X'_{/Y})$ is a field isomorphism. On the other hand, we have just seen that we have an isomorphism $X'_{/Y} \cong X_{/Y}$ between the formal completions which implies that $\widehat{\pi}^*$ is an isomorphism of fields. Since π^* is a field extension of degree $n+1$, we have that $\alpha_{X,Y}$ is a finite field extension of the same degree. That is, Y is $G2$ in X, but not $G3$.

$$\begin{tikzcd} & X' \arrow[d, "\pi"] \\ \mathbb{P}^d \cong Y \arrow[ur, hook, "G3"] \arrow[r, hook, "G2"'] & X. \end{tikzcd}$$

Note also that the normal bundles of Y in X and of Y in X', $N_{Y|X}$ and $N_{Y|X'}$, are both ample and isomorphic to $(n-d)\mathcal{O}_{\mathbb{P}^d}(1)$. In particular, Hartshorne's Theorem 2.5 implies that Y is $G2$ in X and X'.

Remark 3.1. The above example is a generalization of an example of Hartshorne [12], page 440, corresponding to the case $d = 1$ and $n = 3$. Subsequently Hartshorne's example has been extended to the case $d = 1$ and $n \geq 3$ by Bădescu, Beltrametti and Ionescu in [2] to provide one of the non-trivial examples of an almost-line on a manifold of any dimension $n \geq 3$.

4. Results.

The above example lead us to the more general setting: given two embeddings $\mathbb{P}^d \cong Y \hookrightarrow X_i$, $i = 1, 2$, where X_i is a projective n-fold and $N_{Y|X_i} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$, are they *always* formally equivalent? And what can we say about Zariski equivalence?

If $d = 1$, i.e. if Y is a smooth connected curve $Y \cong \mathbb{P}^1$ in a smooth projective variety X with normal bundle $N_{Y|X} \cong (n-1)\mathcal{O}_{\mathbb{P}^1}(1)$ then Y is called a *quasi-line*. The example above shows that there are quasi-lines formally (but not Zariski) equivalent to a line in $\mathbb{P}^n$. Furthermore, there are also examples of quasi-lines in a smooth projective complex variety of dimension $2n-1$ ($n \geq 2$) which are not even formally equivalent to a line in $\mathbb{P}^{2n-1}$. See [2], cf. also [1], Chapter 14, for a thorough analysis of this case.

We first consider the case $d \geq 3$. As a first result we have the following:

Theorem 4.1. *Let $\mathbb{P}^d \cong Y \hookrightarrow X_1$, $\mathbb{P}^d \cong Y \hookrightarrow X_2$, with $d \geq 3$, be two embeddings in two n-folds projective varieties X_1, X_2 with normal bundle $N_{Y|X_1} \cong N_{Y|X_2} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$. Then the two embeddings are formally equivalent i.e. $X_{1/Y} \cong X_{2/Y}$. Moreover, Y is $G2$ in both X_1 and X_2.*

Proof. The key tool of the proof is Corollary 2.9 of Grothendieck's Theorem 2.8. Since $N_{Y|X_i} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$ is ample, we only need to check the cohomological vanishing conditions (1) required in the theorem:

$$H^1(Y, T_Y \otimes S^m(N^*)) = H^1(Y, N \otimes S^{m+1}(N^*)) = 0, \qquad \text{for all } m \geq 1, \tag{2}$$

where $S(N^*) := \bigoplus_{m=1}^{\infty} S^m(N^*)$ is the symmetric $\mathcal{O}_Y$-algebra of the dual N^* of N.

First observe that $S^m(N^*) = \binom{m+n-d-1}{m}\mathcal{O}_{\mathbb{P}^d}(-m)$. Now consider the Euler sequence of $\mathbb{P}^d$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^d} \longrightarrow (d+1)\mathcal{O}_{\mathbb{P}^d}(1) \longrightarrow T_Y \longrightarrow 0.$$

Tensoring it by $S^m(N^*)$ we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^d} \otimes S^m(N^*) \longrightarrow (d+1)\mathcal{O}_{\mathbb{P}^d}(1) \otimes S^m(N^*) \longrightarrow T_Y \otimes S^m(N^*) \longrightarrow 0.$$

Denoting $\mu_m := \binom{m+n-d-1}{m}$, the previous exact sequence can be rewritten as

$$0 \longrightarrow \mu_m \mathcal{O}_{\mathbb{P}^d}(-m) \longrightarrow (d+1)\mu_m \mathcal{O}_{\mathbb{P}^d}(1-m) \longrightarrow \mu_m T_Y \otimes \mathcal{O}_{\mathbb{P}^d}(-m) \longrightarrow 0.$$

Computing the exact sequence of cohomology, we get

$$\begin{gathered} \ldots \to \mu_m(d+1)H^1(Y, \mathcal{O}_{\mathbb{P}^d}(1-m)) \to \\ \mu_m H^1(Y, T_Y \otimes \mathcal{O}_{\mathbb{P}^d}(-m)) \to \mu_m H^2(Y, \mathcal{O}_{\mathbb{P}^d}(-m)) \to \ldots. \end{gathered} \tag{3}$$

Since $H^1(Y, \mathcal{O}_{\mathbb{P}^d}(1-m)) = H^2(Y, \mathcal{O}_{\mathbb{P}^d}(-m)) = 0$, then we get the first vanishing:

$$H^1(Y, \mu_m T_Y \otimes \mathcal{O}_{\mathbb{P}^d}(-m)) = 0,$$

for all $m \geq 1$.

As far as the second vanishings in (1) are concerned, we have:

$$H^1(Y, N \otimes S^{m+1}(N^*)) = \mu_{m+1}(n-d)H^1(Y, \mathcal{O}_{\mathbb{P}^d}(-m)) = 0$$

for all $m \geq 1$. So by Gieseker's Theorem we have $X_{1/Y} \cong X_{2/Y}$, i.e. the two embeddings are formally equivalent.

For the last statement in the Theorem, just apply Hartshorne's result, Theorem 2.5, for $N_{Y|X_1} \cong N_{Y|X_2} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$ is ample and so Y is $G2$ in both X_1 and X_2. □

Theorem 4.1 has some immediate consequences.

Corollary 4.2. *Under the same hypothesis of Theorem 4.1, if furthermore we assume Y is $G3$ in both X_1 and X_2, then the two embeddings are Zariski equivalent. In particular, X_1 and X_2 are rational.*

Proof. We have

$$\begin{tikzcd} & X_1 \\ \mathbb{P}^d \cong Y \arrow[ur, hook, "G3"] \arrow[r, hook, "G3"'] & X_2. \end{tikzcd}$$

By Theorem 4.1, $X_{1/Y} \cong X_{2/Y}$, and thus we can apply Gieseker's Theorem 2.7 to conclude that the two pairs are Zariski equivalent.

Now consider the embedding $Y \hookrightarrow \mathbb{P}^n$. Then by Hironaka-Matsumura Theorem 2.3, Y is $G3$ in $\mathbb{P}^n$ and applying Theorem 4.1, we also have that $(\mathbb{P}^n, Y)$ is formally equivalent to (X_i, Y). Therefore,

$$K(\mathbb{P}^n) \cong K\left(\mathbb{P}^n_{/Y}\right) \cong K(X_{i/Y}) \cong K(X_i),$$

and so X_1 and X_2 are rational. □

Corollary 4.3. *Let* $\mathbb{P}^d \cong Y \overset{i}{\hookrightarrow} X$ *be an embedding of* d*-linear projective space* ($d \geq 3$) *into a* n*-fold projective variety* X*, with* $N_{Y|X} \cong (n-d)\mathcal{O}_{\mathbb{P}^d}(1)$. *Then* X *is unirational.*

Proof. By a Theorem of Hartshorne and Gieseker (see [5], Theorem 4.3, or also [1], Corollary 9.20), there exists a finite, surjective morphism $f : X' \to X$ of degree $[K(X_{/Y}) : K(X)]$, where X' is a normal projective variety, such that the inclusion i lifts to an inclusion $i' : Y \hookrightarrow X'$ such that f is étale in a neighborhood of $i'(Y)$ and $i'(Y)$ is $G3$ in X'. Hence by the previous Corollary, X' is rational and since f is a finite, surjective morphism, X is unirational. □

Remark 4.4. The previous Corollary is valid in a more general setting than Corollary 4.2: in particular it says that under our usual hypothesis about the normal bundle of Y in X and if Y is not $G3$ in X, then X is unirational.

Now we conclude with some remarks concerning the case $d = 2$, which is somewhat more complicated.

According to the proof of Theorem 4.1, we see that the first cohomological vanishings (1) required in Grothendieck's Theorem 2.8 hold only for $d \geq 3$. In fact, assume $d = 2$ and rewrite the cohomological sequence (3):

$$\begin{aligned} 0 &\to \mu_m H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m)) \to \mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m)) \to \\ &\to 3\mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-m)) \to \mu_m H^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m)) \to 0 \end{aligned} \tag{4}$$

For $m = 1$ or $m = 2$, we have $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m)) = 0$. As a consequence $H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m)) = 0$, for $m = 1, 2$.

Let us now compute the dimension of the last cohomology group in the above sequence (4), $h^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m))$: by Serre's Duality, this is equal to $h^0(\mathbb{P}^2, T^*_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m-3))$. By a simple observation of linear algebra, this is equal to $h^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m-6))$. Using the Euler sequence for $\mathbb{P}^2$ and tensorising it with $\mathcal{O}_{\mathbb{P}^2}(m-6)$, we get

$$0 \to \mathcal{O}_{\mathbb{P}^2}(m-6) \to 3\mathcal{O}_{\mathbb{P}^2}(m-5) \to T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m-6) \to 0.$$

Passing to the cohomological sequence we obtain

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-6)) \to 3H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-5)) \to$$
$$\to H^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m-6)) \to 0.$$

In particular,

- $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-6)) = \dfrac{(m-4)(m-5)}{2}$, for $m \geq 4$;
- $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-5)^{\oplus 3}) = \dfrac{3(m-3)(m-4)}{2}$;
- $h^0(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m-6)) = \dfrac{3(m-3)(m-4)}{2} - \dfrac{(m-4)(m-5)}{2} = (m-2)(m-4) = h^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m))$, for $m \geq 4$.

Now we are able to compute $h^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m))$. Using the exact sequence (4), for $m \geq 4$ we get:

$$\begin{aligned} h^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m)) &= h^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-m)) - 3h^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1-m)) + \\ &\quad + h^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m) = \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-3)) - 3h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m-4)) + \\ &\quad + h^2(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-m)) = \\ &= \frac{(m-1)(m-2)}{2} - \frac{3(m-2)(m-3)}{2} \\ &\quad + (m-2)(m-4) = 0. \end{aligned}$$

It follows that the vanishing $H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes S^m(N^*)) = 0$ is verified for $m = 1, 2$ and for $m \geq 4$. If we denote by $Y(m)_i$ the m-th infinitesimal neighbourhood of Y in X_i, $i = 1, 2$, $m \geq 0$, this means that there exists an isomorphism of schemes

$$Y(2)_1 \cong Y(2)_2$$

inducing identity on Y. Moreover, from the forth infinitesimal neighborhood on everything works well. So what does happen in the third infinitesimal neighborhood, i.e. when $m = 3$? Let us compute the cohomological sequence (4) with $m = 3$:

$$0 \to \mu_m H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \to \mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \to$$

$$\to 3\mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) \to \mu_m H^2(\mathbb{P}^2, T^2_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \to 0.$$

Since $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, then

$$0 \to \mu_m H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \xrightarrow{\sim} \mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \to 0,$$

and

$$\dim\left(\mu_m H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))\right) = \dim\left(\mu_m H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3))\right) = \mu_m = \binom{n}{3}.$$

So there is an obstruction in constructing the third infinitesimal neighborhood of Y in X_1 and X_2 such that they are isomorphic.

Assuming that $Y(3)_1 \cong Y(3)_2$ then it immediately follows that the two embeddings are formally equivalent. The viceversa is still true, because of the definition of formal equivalence.

Summing up the above discussion we get the following:

Theorem 4.5. *Let $\mathbb{P}^2 \cong Y \hookrightarrow X_1, \mathbb{P}^2 \cong Y \hookrightarrow X_2$ be two embeddings in two n-folds projective varieties X_1, X_2 with normal bundle $N_{Y|X_i} \cong (n-2)\mathcal{O}_{\mathbb{P}^2}(1), i = 1, 2$. Denote by $Y(j)_i$ the j-th infinitesimal neighborhood of Y in $X_i, i = 1, 2$. Then $Y(2)_1 \cong Y(2)_2$ (inducing identity on Y). Moreover the embeddings $Y \subset X_1$ and $Y \subset X_2$ are formally equivalent if and only if the infinitesimal neighborhoods of order* 3 *of Y in X_1 and in X_2 are isomorphic, i.e. $Y(3)_1 \cong Y(3)_2$ (inducing identity on Y).*

REFERENCES

[1] L. Bădescu, *Projective Geometry and Formal Geometry,* Birkäuser, 2004.

[2] L. Bădescu - M. C. Beltrametti - P. Ionescu, *Almost-lines and quasi-lines on projective manifolds,* in “Complex Analysis and Algebraic Geometry - A Volume in Memory of Michael Schneider” (Th. Peternell, F-O. Schreyer, editors), Walter de Gruyter, Berlin-New York, 2000, pp. 1-27.

[3] L. Bădescu - M. Schneider, *Formal functions, connectivity and homogeneous spaces,* in “Algebraic Geometry - A Volume in Memory of Paolo Francia” (M. C. Beltrametti, F. Catanese, C. Ciliberto, A. Lanteri, C. Pedrini, editors), Walter de Gruyter, Berlin-New York, 2002, pp. 1-21.

[4] G. Faltings, *Formale Geometrie und homogene Räume,* Invent. Math. 64 (1981), pp. 123-165.

[5] D. Gieseker, *On two theorems of Griffiths about embeddings with ample normal bundle,* Amer. J. Math. 99, 1977, pp. 1137-1150.

[6] A. Grothendieck, *Cohomologie Locale des Faiceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux,* North-Holland, Amsterdam, 1968.

[7] A. Grothendieck, *Revêtement Étales et Groupe Fondamental,* Lect. Notes in Math., Vol. 224, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[8] A. Grothendieck - J. Diedonné, *Éléments de Géométrie Algébrique,* Publ. Math. IHES, Vol. 4, 8, 11, 17, 20, 24, 28, 32, Bures-sur-Yvette, 1960-1967.

[9] R. Hartshorne, *Curves with high self-intersection on an algebraic surface,* Publ. Math. IHES, Vol. 36, 1969, pp. 11-126.

[10] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties,* Lect. Notes in Math., Vol. 156, Springer-Verlag, 1970.

[11] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Math., Springer-Verlag, 1977.

[12] R. Hartshorne, *Cohomological dimensional of algebraic varieties,* Annals of Math. 88 (1968), pp. 403-450.

[13] H. Hironaka - H. Matsumura, *Formal functions and formal embeddings,* J. Math. Soc. Japan 20 (1968), pp. 52-82.

[14] O. Zariski, *Theory and Applications of Holomorphic Functions on Algebraic Varieties over Arbitrary Ground Fields,* Memoirs of the American Math. Soc., Vol. 5, 1951.

Francesca Incensi
Università degli Studi di Bologna
Dipartimento di Matematica
Piazza di Porta San Donato
40126 Bologna, Italy
e-mail: incensi@dm.unibo.it

Helena Soares
Universitat de Barcelona
Facultat de Matemàtiques
Department d'Àlgebra and Geometria
Gran Via de les Corts Catalanes 585
08007 Barcelona, Spain
e-mail: helena.soares@ub.edu

ISCTE, Escola de Gestão
Departamento de Métodos Quantitativos
Av. Forças Armadas
1649-026 Lisbon, Portugal
e-mail: helena.soares@iscte.pt