EXCEPTIONAL BUNDLES OF HOMOLOGICAL DIMENSION k

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ABSTRACT. We characterize exceptional vector bundles on $\mathbb{P}^n$ of arbitrary homological dimension defined by a linear resolution. Moreover, we determine all Betti numbers of such resolution.

1. Introduction

Let E be a vector bundle on $\mathbb{P}^n$. A resolution of E is an exact sequence

$$0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to E \to 0,$$

where F_i splits as a direct sum of line bundles. The minimal length of such resolutions is called the *homological dimension of* E and it is denoted by $\mathrm{hd}(E)$. Besides, it is well-known that $\mathrm{hd}(E) \leq n-1$.

We say that the resolution of E is *linear* if it is of the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-k)^{a_k} \xrightarrow{\alpha_k} \cdots \xrightarrow{\alpha_2} \mathcal{O}_{\mathbb{P}^n}(d-1)^{a_1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}(d)^{a_0} \to E \to 0,$$

for some $d \in \mathbb{Z}$, i.e. the entries of the matrices associated to the morphisms α_i are linear forms. The exponents a_i are called the *Betti numbers of* E.

In the present work we will focus our attention on vector bundles E on the projective space $\mathbb{P}^n$ with $\mathrm{hd}(E) \leq k$, for some $k \in \{1, \ldots, n-1\}$ a positive integer, and defined by a linear resolution of type

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k)^{a_k} \xrightarrow{\alpha_k} \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \xrightarrow{\alpha_{k-1}} \cdots \xrightarrow{\alpha_2} \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}^{a_0} \xrightarrow{\alpha_0} E \longrightarrow 0. \quad (1)$$

Our purpose is to characterize those E which are exceptional, i.e whose only endomorphisms are the homotheties and satisfying $\mathrm{Ext}^q(E,E) = 0$, for all $q \geq 1$. Exceptional bundles are a powerful tool in the study of stable sheaves and the derived category of coherent sheaves. They first appeared in a paper by Drézet and Le Potier [DLP85], who used them to describe the possible ranks and Chern classes of semi-stable vector bundles on $\mathbb{P}^2$. Their study was then formulated within the setting of derived

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categories, due to mathematicians such as Gorodentsev and Kuleshov [GK04]. This gave rise to a technique called Helix Theory whose main idea is to describe the set of exceptional bundles over a variety and to produce new ones by means of mutations.

The first non-trivial example of vector bundles on $\mathbb{P}^n$ of type (1) is the case of homological dimension 1, that is when E is defined by a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0.$$

These bundles were introduced by Dolgachev and Kapranov [DK93] and they are usually called Steiner bundles. Herein we will call them *classical Steiner bundles* and will consider them as a particular case of the more general notion of a Steiner bundle on algebraic varieties (see Definition 2.2).

The study of classical Steiner bundles is extensive and has been addressed from very different points of view. With respect to the scope of this paper we refer the work of Brambilla and Soares ([Bra04], [Bra08] and [Soa08]). The first proved that any exceptional general classical Steiner bundle E is characterised by $\chi(\operatorname{End} E) = 1$. In the same spirit, Soares [Soa08] introduced an appropriate generalization of the concept of a Steiner bundle on an algebraic variety X and showed that an exceptional Steiner bundle E on X is also characterized by $\chi(\operatorname{End} E) = 1$, provided E is general. In both cases, a complete description of the resolution of E is provided.

It is a natural step to work towards the characterization of exceptional bundles E of higher homological dimension (we refer to [MP14], where this problem is also discussed). So in section 2 we first recall the main concepts and results on vector bundles on algebraic varieties of homological dimension 1 with a linear resolution, the so-called Steiner bundles, needed later on.

We address the characterization of exceptional bundles of arbitrary homological dimension with a resolution of type (1) in section 3. Such resolution splits into k short exact sequences. Denoting $S_i = \operatorname{coker} \alpha_{i+1}$, $i = 1, \dots, k-1$, the farther vector bundle S_{k-1} is a Steiner bundle, so we know exactly when it is exceptional. This fact will clearly play a role in achieving our goal.

Since any exceptional bundle F satisfies $\chi(\operatorname{End} F) = 1$ we compute the Euler characteristic of $\operatorname{End} E$. We get an iterative formula that interrelates the Euler characteristics $\chi(\operatorname{End} E)$ and $\chi(\operatorname{End} S_i)$ (Proposition 3.3) and we compute $\chi(\operatorname{End} E)$ in terms of the Betti numbers of E (Proposition 3.4). Our first important result (Proposition 3.5) will give sufficient conditions so that $\chi(\operatorname{End} S_i) = 1$, for all $i = 1, \dots, k-1$, and it will be the main tool to prove our main theorem (Theorem 3.7). We prove that a vector bundle E defined by (1) is exceptional if and only if $\chi(\operatorname{End} E) = 1$ and all S_i are exceptional, provided we impose a cohomological vanishing condition on E. Furthermore, if this is the this case, we are able to describe all Betti numbers in (1). They are completely determined by the exponents that define the resolution of the exceptional Steiner bundle S_{k-1} (see Corollary 3.8).

In the last section, section 4, we give some examples of exceptional bundles E of homological dimension $k \geq 2$. For instance, a theorem by Ellia and Hirchowitz (see Theorem 4.1) will help us to explicitly construct a large family of exceptional vector bundles of homological dimension 2. In the case when $\mathrm{hd}(E) \geq 3$ we provide several examples with the help of Macaulay2 [GS] which give support to the conjecture stated in the end of the paper.

Notation 1.1. Given a smooth algebraic variety X of dimension n and a coherent sheaf E on X, we denote $\mathrm{H}^q_*(X, E) = \oplus_{m\in\mathbb{Z}} \mathrm{H}^q(X, E(m))$, $\mathrm{H}^q(E(m)) = \mathrm{H}^q(X, E(m))$ and $\mathrm{h}^q(E) = \dim \mathrm{H}^q(E)$. The Euler characteristic of E is defined by the integer $\chi(E) = \sum_{i=0}^n (-1)^i \mathrm{h}^i(E)$.

Recall furthermore that when E is a locally free sheaf, we have an isomorphism $\operatorname{End} E \cong E^\vee \otimes E$.

Throughout this paper we will consider K a fixed algebraically closed field of characteristic 0.

2. Exceptional vector bundles of homological dimension 1

Let E be a vector bundle on $\mathbb{P}^n$, $n \geq 3$, with a linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_k} \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0,$$

so that $\mathrm{hd}(E) \leq k$. When $\mathrm{hd}(E) = 1$, E is defined by a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0,$$

and we will call E a *classical Steiner bundle*.

In this section we recall the main results which we will be using throughout the present paper on vector bundles of homological dimension 1. Although we will only deal with vector bundles on the projective space, we introduce all concepts and state their properties in the broadest setting.

Definition 2.1. Let X be a smooth algebraic variety. A coherent sheaf E on X is called *simple* if $\operatorname{Hom}(E, E) \simeq K$. If, furthermore, it satisfies

$$\operatorname{Ext}^q(E, E) = 0, \text{ for all } q \geq 1,$$

then we say that E is *exceptional.*

An ordered pair (E, F) of coherent sheaves on X is *strongly exceptional* if both E and F are exceptional and

$$\operatorname{Ext}^q(F, E) = 0, \quad \forall\, q \geq 0,$$
$$\operatorname{Ext}^q(E, F) = 0, \quad \forall\, q \neq 0.$$

In [Bra08], Brambilla characterized simple and exceptional classical Steiner bundles. Her work was generalised in [Soa08] to vector bundles on smooth irreducible algebraic varieties. So, a natural definition of Steiner bundles in this general context was introduced:

Definition 2.2. A vector bundle E on a smooth irreducible algebraic variety X is called a *Steiner bundle* if it is defined by an exact sequence of the form

$$0 \to F_0^s \to F_1^t \to E \to 0, \tag{2}$$

where $s, t \geq 1$ and (F_0, F_1) is an ordered pair of vector bundles on X satisfying the following two conditions:

(i) (F_0, F_1) is strongly exceptional;
(ii) $F_0^\vee \otimes F_1$ is generated by its global sections.

It is immediate from Definition 2.1 that any exceptional vector bundle E satisfies $\chi(\operatorname{End} E) = 1$. If E is a Steiner bundle on X then the Euler characteristic of $\operatorname{End} E$ has a very simple formula and it is possible to describe all solutions of the equation $\chi(\operatorname{End} E) = 1$.

Lemma 2.3 ([Soa08], Lemma 2.2.3)**.** *Let E be a Steiner bundle on a smooth irreducible algebraic variety X defined by the exact sequence* (2) *and let $\lambda = \mathrm{h}^0(F_0^\vee \otimes F_1)$. Then*

$$\chi(\operatorname{End} E) = t^2 + s^2 - \lambda st.$$

Moreover,

$$\chi(\operatorname{End} E) = 1 \Leftrightarrow \exists\, k \in \mathbb{N} \;\; s.\ t.\;\; s = u_k,\;\; t = u_{k+1},$$

where $\{u_k\}_{k\geq 0}$ is the sequence defined recursively by

$$\begin{cases} u_0 = 0 \\ u_1 = 1 \\ u_{k+1} = \lambda u_k - u_{k-1}. \end{cases} \tag{3}$$

Remark 2.4. We say that a vector bundle E on X fitting in a short exact sequence of the form

$$0 \to F_0^s \xrightarrow{m} F_1^t \to E \to 0,$$

is general if m is general in the affine space $\operatorname{Hom}(F_0^s, F_1^t) \simeq K^s \otimes K^t \otimes K^\lambda$, where $\lambda = h^0((F_0)^\vee \otimes F_1)$.

As we mentioned above, the Euler characteristic of $\operatorname{End} E$ of any exceptional vector bundle E is always 1. The converse is not true in general. In spite of that, in the case of Steiner bundles the equation $\chi(\operatorname{End} E) = 1$ indeed characterizes exceptional bundles, provided E is general (Remark 2.4).

Theorem 2.5 ([Soa08], Theorem 2.2.7). *Let E be a general Steiner bundle on a smooth irreducible algebraic variety X defined by an exact sequence of type* (2). *Assume $\lambda = h^0(F_0^\vee \otimes F_1) \geq 3$. Then*

$$E \text{ is exceptional if and only if } \chi(\operatorname{End} E) = 1.$$

Equivalently, E is exceptional if and only if it is of the form

$$0 \to F_0^{u_k} \to F_1^{u_{k+1}} \to E \to 0,$$

for some $k \in \mathbb{N}$, where $\{u_k\}_{k\geq 1}$ is the sequence in (3).

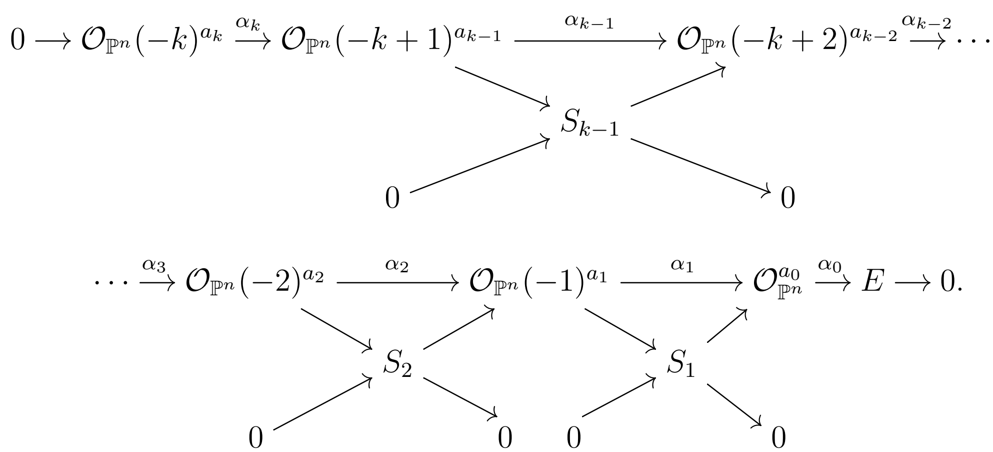
3. Exceptional vector bundles of homological dimension k

The main purpose in this section is to study exceptional bundles of arbitrary homological dimension and to generalize in some way the results of the previous section on vector bundles of homological dimension 1.

Let E be a vector bundle on $\mathbb{P}^n$, $n \geq 3$, with linear resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k)^{a_k} \xrightarrow{\alpha_k} \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \xrightarrow{\alpha_{k-1}} \cdots \xrightarrow{\alpha_2} \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}^{a_0} \xrightarrow{\alpha_0} E \longrightarrow 0. \quad (4)$$

In particular, $\operatorname{hd}(E) \leq k \leq n-1$. Cut this long exact sequence by setting $S_i = \operatorname{coker} \alpha_{i+1} = \ker \alpha_{i-1}$, for $i = 1, \dots, k-1$, that is



(5)

We thus get the following k short exact sequences:

$$\begin{gathered} 0 \to \mathcal{O}_{\mathbb{P}^n}(-k)^{a_k} \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \to S_{k-1} \to 0, \\ 0 \to S_{k-1} \to \mathcal{O}_{\mathbb{P}^n}(-k+2)^{a_{k-2}} \to S_{k-2} \to 0, \\ \vdots \\ 0 \to S_2 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to S_1 \to 0, \\ 0 \to S_1 \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0. \end{gathered} \quad (6)$$

Set $S_0 = E$, so we can write more generally $S_i = \operatorname{coker} \alpha_{i+1}$, $i = 0, \dots, k-1$. We start with two lemmas regarding some cohomological properties of the vector bundles E and S_i that will be useful in the sequel.

Lemma 3.1. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution* (4) *and let $S_i = \operatorname{coker} \alpha_{i+1}$, $i = 0, \dots, k-1$. Then*

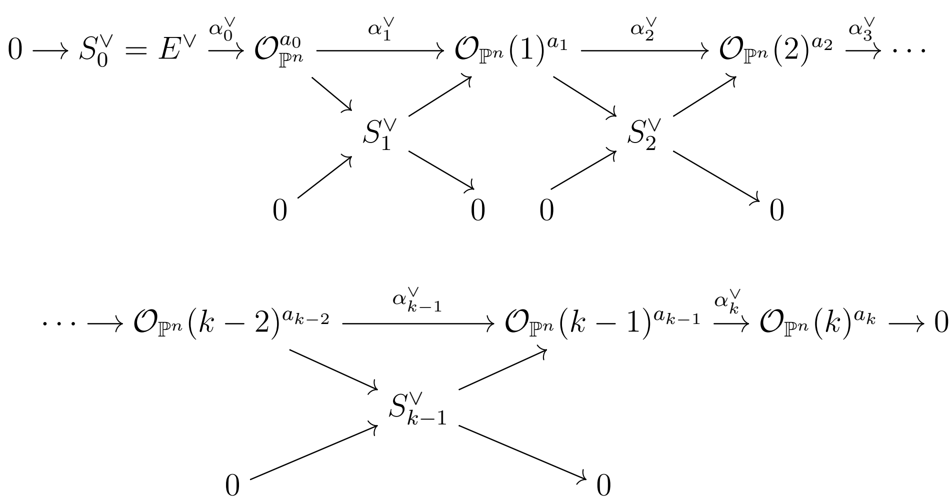
$$\mathrm{H}^q_*(S_i) = 0, \quad \forall\, q \neq 0, n-k+i, n.$$

Proof. First observe that $\operatorname{hd}(E) = \operatorname{hd}(S_0) \leq k$ and $\operatorname{hd}(S_i) \leq k - i$, for each $i = 1, \dots, k-1$. So, recalling Proposition 1.4 in [BS92], we have

$$\mathrm{H}^q_*(S_i) = 0, \quad \forall\, q = 1, \dots, n-k+i-1, \tag{7}$$

for each $i = 0, \dots, k-1$.

Consider the dual resolution of (5):



From $\operatorname{hd}\left(S_{k-1}^\vee\right) \leq n-1$ it follows that $\operatorname{hd}\left(S_i^\vee\right) \leq n-k+i$. Hence, Proposition 1.4 in [BS92] also gives us

$$\mathrm{H}^q_*\left(S_i^\vee\right) = 0, \quad \forall\, q = 1, \dots, k-i-1, \tag{8}$$

for each $i = 0, \dots, k-1$.

Applying Serre duality to conditions (8), together with equalities (7), we obtain

$$\mathrm{H}^q_*(S_i) = 0, \quad \forall\, q \neq 0, n-k+i, n.$$

□

Lemma 3.2. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution* (4)*. Let $S_i = \operatorname{coker} \alpha_{i+1}$, $i = 0, \dots, k-1$. Then, for any $m \in \mathbb{Z}$:*

$$\mathrm{h}^0(S_{k-1}(m)) = a_{k-1}\binom{n+m-k+1}{m-k+1} - a_k\binom{n-k+m}{-k+m},$$
$$\mathrm{h}^{n-1}(S_{k-1}(m)) = a_k\binom{-m+k-1}{-m-n+k-1} - a_{k-1}\binom{-m+k-2}{-m-n+k-2} + \mathrm{h}^n(S_{k-1}(m)),$$

and, for $i = 0, \dots, k-2$,

$$\mathrm{h}^0(S_i(m)) = a_i\binom{n+m-i}{m-i} - \mathrm{h}^0\left(S_{i+1}(m)\right),$$
$$\mathrm{h}^{n-k+i}(S_i(m)) = \mathrm{h}^{n-k+i+1}(S_{i+1}(m)),$$
$$\mathrm{h}^n(S_i(m)) = a_i\binom{-m+i-1}{-m-n+i-1} - \mathrm{h}^n(S_{i+1}(m)).$$

Proof. It follows from the previous lemma and from applying cohomology to the short exact sequences (6). □

Amongst the set of vector bundles with resolution of type (4) we are interested in studying those which are exceptional. Computing $\chi(\operatorname{End} E)$ is thus a natural step. We present two formulas for the Euler characteristic of $\operatorname{End} E$. The first is an iterative formula and it will be especially useful in the proof of Proposition 3.5.

Proposition 3.3. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution (4) and let $S_i = \operatorname{coker}\alpha_{i+1}$, $i = 0, \dots, k-1$. Then, for all $i = 0, \dots, k-2$,*

$$\chi(\operatorname{End} S_i) = \chi\left(\operatorname{End} S_{i+1}\right) + a_i\left(a_i - \chi\left(S_{i+1}^{\vee}(-i)\right)\right), \tag{9}$$

and

$$\chi(\operatorname{End} S_{k-1}) = a_{k-1}^2 + a_k^2 - (n+1)a_{k-1}a_k.$$

In particular,

$$\chi(\operatorname{End} E) = \chi\left(\operatorname{End} S_i\right) + a_{i-1}\left(a_{i-1} - \chi\left(S_i^{\vee}(-i+1)\right)\right) + a_{i-2}\left(a_{i-2} - \chi\left(S_{i-1}^{\vee}(-i+2)\right)\right)$$
$$+ \dots + a_1\left(a_1 - \chi\left(S_2^{\vee}(-1)\right)\right) + a_0\left(a_0 - \chi\left(S_1^{\vee}\right)\right). \tag{10}$$

Proof. We first note that according to Definition 2.2 the vector bundle S_{k-1} is a Steiner bundle on $\mathbb{P}^n$, so we know by Lemma 2.3 that

$$\chi(\operatorname{End} S_{k-1}) = a_{k-1}^2 + a_k^2 - (n+1)a_{k-1}a_k.$$

We need to prove (9), that is,

$$\chi(\operatorname{End} S_i) = \chi\left(\operatorname{End} S_{i+1}\right) + a_i\left(a_i - \chi\left(S_{i+1}^{\vee}(-i)\right)\right),$$

for every $i = 0, \dots, k-2$. This will immediately imply (10). We will prove the statement by induction on k.

If $k = 1$ then $E = S_0 = S_{k-1}$ is a classical Steiner bundle on $\mathbb{P}^n$ with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0$$

and we already saw that

$$\chi(\operatorname{End} E) = a_0^2 + a_1^2 - (n+1)a_0a_1.$$

Now suppose that the statement holds for every vector bundle F on $\mathbb{P}^n$ with a linear resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \to \mathcal{O}_{\mathbb{P}^n}(-k+2)^{a_{k-2}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to F \to 0,$$

and let us prove it for any vector bundle E of homological dimension at most k and linear resolution of type (4). If E is such a vector bundle then $S_1(1)$ is defined by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_k} \to \mathcal{O}_{\mathbb{P}^n}(-k+2)^{a_{k-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{a_1} \to S_1(1) \to 0.$$

Hence $\operatorname{hd}(S_1(1)) \leq k-1$ and by the induction hypothesis we know that, for any $i = 1, \ldots, k-2$,

$$\begin{aligned} \chi(\operatorname{End} S_i(1)) &= \chi\left(\operatorname{End} S_{i+1}(1)\right) + a_i\left(a_i - \chi\left(S_{i+1}(1)^\vee(-i+1)\right)\right) \\ &= \chi\left(\operatorname{End} S_{i+1}(1)\right) + a_i\left(a_i - \chi\left(S_{i+1}^\vee(-i)\right)\right), \end{aligned}$$

or equivalently,

$$\chi(\operatorname{End} S_i) = \chi\left(\operatorname{End} S_{i+1}\right) + a_i\left(a_i - \chi\left(S_{i+1}^\vee(-i)\right)\right).$$

So the only case left to prove is when $i = 0$. Dualising and twisting by E the last short exact sequence in (6) we get

$$0 \to \operatorname{End} E \cong E^\vee \otimes E \to E^{a_0} \to S_1^\vee \otimes E \to 0$$

and hence,

$$\chi\left(\operatorname{End} E\right) = a_0\chi(E) - \chi\left(S_1^\vee \otimes E\right).$$

On the other hand, from the same sequence in (6) we deduce that

$$\chi(E) = a_0\chi\left(\mathcal{O}_{\mathbb{P}^n}\right) - \chi(S_1) = a_0 - \chi(S_1),$$

and also (twisting it by $S_1^\vee$)

$$\chi\left(S_1^\vee \otimes E\right) = a_0\chi\left(S_1^\vee\right) - \chi\left(S_1^\vee \otimes S_1\right) = a_0\chi\left(S_1^\vee\right) - \chi\left(\operatorname{End} S_1\right).$$

Therefore,

$$\chi\left(\operatorname{End} E\right) = a_0^2 - a_0\chi(S_1) - a_0\chi\left(S_1^\vee\right) + \chi\left(\operatorname{End} S_1\right).$$

Now, applying Lemma 3.1 to S_1, we know that

$$\chi(S_1) = \mathrm{h}^0(S_1) + (-1)^{n-k+1}\mathrm{h}^{n-k+1}(S_1) + (-1)^n\mathrm{h}^n(S_1).$$

But from Lemma 3.2 and (6) we compute

$$\mathrm{h}^0(S_1) = 0, \quad \mathrm{h}^n(S_1) = 0, \quad \mathrm{h}^{n-k+1}(S_1) = \mathrm{h}^{n-1}(S_{k-1}) = 0.$$

So $\chi(S_1) = 0$ and

$$\chi(\operatorname{End} E) = \chi(\operatorname{End}(S_1)) + a_0 \left(a_0 - \chi(S_1^\vee)\right),$$

which completes the proof of (9). □

An alternative formula for $\chi(\operatorname{End} E)$ can be given depending only on the Betti numbers of E.

Proposition 3.4. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution* (4)*. Then*

$$\begin{aligned}\chi(\operatorname{End} E) &= \sum_{i=0}^{k} a_i^2 - (n+1)\sum_{i=0}^{k-1} a_i a_{i+1} + \binom{n+2}{2}\sum_{i=0}^{k-2} a_i a_{i+2} - \cdots + \\ &+(-1)^{k-1}\binom{n+k-1}{k-1}(a_0 a_{k-1} + a_1 a_k) + (-1)^k \binom{n+k}{k} a_0 a_k.\end{aligned}$$

Proof. We will prove the statement by induction on k. If $k = 1$ then E is a classical Steiner bundle on $\mathbb{P}^n$ and according to Lemma 2.3 we have

$$\chi(\operatorname{End} E) = a_0^2 + a_1^2 - (n+1)a_0 a_1.$$

Suppose that every vector bundle F with a linear resolution of type

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_{k-1}} \to \mathcal{O}_{\mathbb{P}^n}(-k+2)^{a_{k-2}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_1} \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to F \to 0.$$

satisfies

$$\begin{aligned}\chi(\operatorname{End} F) &= \sum_{i=0}^{k-1} a_i^2 - (n+1)\sum_{i=0}^{k-2} a_i a_{i+1} + \binom{n+2}{2}\sum_{i=0}^{k-3} a_i a_{i+2} - \cdots + \\ &+(-1)^{k-1}\binom{n+k-1}{k-1} a_0 a_{k-1}.\end{aligned}$$

Let E be a vector bundle on $\mathbb{P}^n$ defined by (4) and let S_i be the vector bundles obtained by cutting the resolution of E into short exact sequences as in (6). In particular, $S_1(1)$ is a vector bundle defined by

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-k+1)^{a_k} \to \mathcal{O}_{\mathbb{P}^n}(-k+2)^{a_{k-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{a_1} \to S_1(1) \to 0.$$

Applying the induction hypothesis to $S_1(1)$ we get

$$\begin{aligned}\chi(\operatorname{End} S_1) = \chi(\operatorname{End} S_1(1)) &= \sum_{i=1}^{k} a_i^2 - (n+1)\sum_{i=1}^{k-1} a_i a_{i+1} + \binom{n+2}{2}\sum_{i=1}^{k-2} a_i a_{i+2} - \cdots + \\ &+(-1)^{k-1}\binom{n+k-1}{k-1} a_1 a_k.\end{aligned}$$

The previous proposition gives us a formula tying $\chi(\operatorname{End} E)$ in with $\chi(\operatorname{End} S_1)$:

$$\chi\left(\operatorname{End} E\right) = \chi\left(\operatorname{End} S_1\right) + a_0^2 - a_0 \chi\left(S_1^\vee\right).$$

So we need to compute $\chi\left(S_1^\vee\right) = (-1)^n\chi\left(S_1(-n-1)\right)$. We have $\mathrm{h}^0(S_1(-n-1)) = 0$ and hence, applying both Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}\chi(S_1(-n-1)) &= (-1)^{n-k+1}\mathrm{h}^{n-k+1}(S_1(-n-1)) + (-1)^n\mathrm{h}^n(S_1(-n-1))\\ &= (-1)^n\left(a_1(n+1) - a_2\binom{n+2}{2} + \cdots + (-1)^{k-1}a_k\binom{n+k}{k}\right).\end{aligned}$$

Therefore,

$$\chi\left(S_1(-n-1)\right) = (-1)^{n-k+1}\mathrm{h}^{k-1}\left(S_1^\vee\right) + (-1)^n\mathrm{h}^0\left(S_1^\vee\right) = (-1)^n\chi\left(S_1^\vee\right)$$

and thus

$$\chi\left(S_1^\vee\right) = a_1(n+1) - a_2\binom{n+2}{2} + \cdots + (-1)^{k-1}a_k\binom{n+k}{k}.$$

At last, we get

$$\begin{aligned}\chi\left(\operatorname{End} E\right) &= \chi\left(\operatorname{End} S_1\right) + a_0^2 - a_0\chi\left(S_1^\vee\right)\\ &= \sum_{i=1}^{k} a_i^2 - (n+1)\sum_{i=1}^{k-1} a_ia_{i+1} + \binom{n+2}{2}\sum_{i=1}^{k-2} a_ia_{i+2} +\\ &\quad + \cdots + (-1)^{k-2}\binom{n+k-2}{k-2}(a_1a_{k-1} + a_2a_k) + (-1)^{k-1}\binom{n+k-1}{k-1}a_1a_k\\ &\quad + a_0^2 - a_0a_1(n+1) + a_0a_2\binom{n+2}{2} + \cdots + (-1)^k a_0a_k\binom{n+k}{k}\\ &= \sum_{i=0}^{k} a_i^2 - (n+1)\sum_{i=0}^{k-1} a_ia_{i+1} + \binom{n+2}{2}\sum_{i=0}^{k-2} a_ia_{i+2} + \cdots +\\ &\quad + (-1)^{k-1}\binom{n+k-1}{k-1}(a_0a_{k-1} + a_1a_k) + (-1)^k\binom{n+k}{k}a_0a_k.\end{aligned}$$

□

We now state the main result that will allow us to achieve a characterization of exceptional bundles which have a linear resolution of length $k \geq 2$. Note that the case of homological dimension 1 was already characterized, as recalled in Theorem 2.5.

Proposition 3.5. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution* (4) *and let $S_i = \operatorname{coker}\alpha_{i+1}$, $i = 1, \ldots, k-1$, with $k \geq 2$.*

Suppose that E is simple, $\chi(\operatorname{End} E) = 1$ and $\mathrm{H}^{n-k}(E(k-3-n)) = 0$. Then $\chi(\operatorname{End} S_i) = 1$, for all $i = 1, \ldots, k-1$.

Proof. We claim that the hypothesis

$$\mathrm{H}^{n-k}(E(k-3-n)) = \mathrm{H}^{n-1}(S_{k-1}(k-3-n)) = \mathrm{H}^1(S_{k-1}^\vee(-k+2)) = 0$$

implies that $S_{k-1}^\vee(-k+3)$ is 0-regular. In fact, we have

$$\mathrm{H}^1(S_{k-1}^\vee(-k+2)) = 0,$$
$$\mathrm{H}^n(S_{k-1}^\vee(-k+3-n)) = 0,$$

where the second vanishing can be obtained from the Serre duality

$$\mathrm{H}^n(S_{k-1}^\vee(-k+3-n)) \cong \mathrm{H}^0(S_{k-1}(k-4))$$

and the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-4)^{a_k} \to \mathcal{O}_{\mathbb{P}^n}(-3)^{a_{k-1}} \to S_{k-1}(k-4) \to 0.$$

The other cohomology groups, $\mathrm{H}^q(S_{k-1}^\vee(-k+3-q))$, $q>0$, all vanish by Lemma 3.1.

Therefore, the vector bundle $S_{k-1}^\vee(-k+3)$ is m-regular for every $m \geq 0$ and in particular we have

$$\mathrm{H}^1\left(S_{k-1}^\vee(m-k+2)\right) = \mathrm{H}^{n-1}\left(S_{k-1}(-m+k-3-n)\right) = 0, \quad \forall m \geq 0. \tag{11}$$

By Proposition 3.3, we know that

$$\chi(\mathrm{End}\, S_{i-1}) = \chi(\mathrm{End}\, S_i) + a_{i-1}\left(a_{i-1} - \chi\left(S_i^\vee(-i+1)\right)\right),$$

where $i = 1, \dots, k-1$. Our goal is to show that

$$\chi\left(S_i^\vee(-i+1)\right) = a_{i-1},$$

for every $i = 1, \dots, k-1$. This will imply

$$\chi(\mathrm{End}\, S_{i-1}) = \chi(\mathrm{End}\, S_i), \quad \forall\, i = 1, \dots, k-1,$$

and, since we are supposing $\chi(\mathrm{End}\, E) = \chi(\mathrm{End}\, S_0) = 1$, we then may conclude that

$$\chi(\mathrm{End}\, S_i) = 1, \quad \forall\, i = 1, \dots, k-1,$$

completing the proof of the proposition.

Let us first prove that $\chi\left(S_1^\vee\right) = a_0$. From the sequence

$$0 \to S_1 \to \mathcal{O}_{\mathbb{P}^n}^{a_0} \to E \to 0,$$

we can easily compute $\mathrm{h}^0(S_1(-n-1)) = 0$. Moreover, it follows from Lemma 3.2 and (11), with $m = k-2 \geq 0$, that

$$\mathrm{h}^{n-k+1}(S_1(-n-1)) = \mathrm{h}^{n-1}(S_{k-1}(-n-1)) = 0.$$

Now, observe that from the above short exact sequence we also get $\mathrm{h}^0(E) = a_0 \neq 0$, for $\mathrm{H}^1(S_1) = 0$ by Lemma 3.1. Due to the hypothesis that E is simple we must have $\mathrm{h}^0\left(E^\vee\right) = \mathrm{h}^n(E(-n-1)) = 0$ (Lemma 4.1.3 in [OSS80]). In particular, this implies that $\mathrm{h}^n(S_1(-n-1)) = a_0$. Applying Lemma 3.1, it holds

$$\chi(S_1(-n-1)) = (-1)^n \mathrm{h}^n(S_1(-n-1)) = (-1)^n \mathrm{h}^0\left(S_1^\vee\right) = (-1)^n \chi\left(S_1^\vee\right) = (-1)^n a_0,$$

hence $\chi\left(S_1^\vee\right) = a_0$.

For $i \geq 2$, consider the sequence

$$0 \to S_i \to \mathcal{O}_{\mathbb{P}^n}(-i+1)^{a_{i-1}} \to S_{i-1} \to 0. \tag{12}$$

Clearly, $\mathrm{h}^0(S_i(i-n-2)) = 0$. On the other hand, applying (11) with $m = k-1-i \geq 0$, we get

$$\mathrm{H}^{n-k+i}(S_i(i-n-2)) = \mathrm{H}^{n-1}\left(S_{k-1}(i-n-2)\right)) = 0.$$

We next compute $\mathrm{H}^n(S_i(i-n-2))$. The cohomology sequence of (12) gives us $\mathrm{h}^n(S_i(i-n-2)) = a_{i-1}\mathrm{h}^n\left(\mathcal{O}_{\mathbb{P}^n}(-n-1)\right) - \mathrm{h}^n(S_{i-1}(i-n-2))$. But analising the sequence

$$0 \to S_{i-2}^{\vee}(-i+1) \to \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{i-2}} \to S_{i-1}^{\vee}(-i+1) \to 0,$$

we see that $\mathrm{h}^n(S_{i-1}(i-n-2)) = \mathrm{h}^0\left(S_{i-1}^{\vee}(-i+1)\right) = 0$. Hence, $\mathrm{h}^n(S_i(i-n-2)) = a_{i-1}$ and thus

$$\chi\left(S_i^{\vee}(-i+1)\right) = \chi\left(S_i(-i-n-2)\right) = a_{i-1},$$

as required.

□

Remark 3.6. Consider the equalities

$$\mathrm{H}^{n-k}(E(k-3-n)) = \mathrm{H}^{n-1}(S_{k-1}(k-3-n)) = \mathrm{H}^1\left(S_{k-1}^{\vee}(-k+2)\right)$$

and the sequence

$$0 \to S_{k-1}^{\vee}(-k+2) \to \mathcal{O}_{\mathbb{P}^n}(1)^{a_{k-1}} \to \mathcal{O}_{\mathbb{P}^n}(2)^{a_k} \to 0.$$

One gets from cohomology the exact sequence

$$0 \to \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^n}(1)\right)^{a_{k-1}} \to \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^n}(2)\right)^{a_k} \to \mathrm{H}^1\left(S_{k-1}^{\vee}(-k+2)\right) \to 0.$$

Thus the vanishing hypothesis $\mathrm{H}^1\left(S_{k-1}^{\vee}(-k+2)\right) = \mathrm{H}^{n-k}(E(k-3-n)) = 0$ in Proposition 3.5 is equivalent to the surjectivity of $\mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^n}(1)\right)^{a_{k-1}} \to \mathrm{H}^0\left(\mathcal{O}_{\mathbb{P}^n}(2)\right)^{a_k}$.

It turns out that this condition is not too restrictive due to Ellia and Hirschowitz's theorem (see Theorem 4.1 in section 4): given a general morphism

$$\mathcal{O}_{\mathbb{P}^n}(1)^{a_{k-1}} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(2)^{a_k},$$

its kernel is a globally generated vector bundle with natural cohomology. Especially, the corresponding morphism $\mathrm{H}(\phi)$ in cohomology is surjective.

We are now able to state our main theorem which under a certain cohomological assumption characterizes exceptional vector bundles of any homological dimension.

Theorem 3.7. *Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution* (4) *and let $S_i = \operatorname{coker}\alpha_{i+1}$, $i = 1, \ldots, k-1$, with $k \geq 2$. Suppose that $\mathrm{H}^{n-k}(E(k-3-n)) = 0$. Then:*

(a) *If E is exceptional then S_i is exceptional, for all $i = 1, \ldots, k-1$.*

(b) *If* $\chi(\operatorname{End} E) = 1$ *and* S_i *is exceptional, for* $i = 1, \dots, k-1$, *then* E *is exceptional.*

Proof. We first prove (a), so suppose that $\mathrm{H}^{n-k}(E(k-3-n)) = 0$ and E is exceptional. Set $S_0 = E$ and consider the sequence

$$0 \to S_i \to \mathcal{O}_{\mathbb{P}^n}(-i+1)^{a_{i-1}} \to S_{i-1} \to 0, \tag{13}$$

with $i = 1, \dots, k-1$. Dualise and twist this sequence by S_i, so we get

$$0 \to S_i \otimes S_{i-1}^{\vee} \to S_i(i-1)^{a_{i-1}} \to S_i \otimes S_i^{\vee} \to 0. \tag{14}$$

Lemma 3.2 gives us

$$\begin{aligned} \mathrm{h}^0(S_i(i-1)) &= \mathrm{h}^0(S_{k-1}(i-1)) = 0, \\ \mathrm{h}^{n-k+i}(S_i(i-1)) &= \mathrm{h}^{n-1}(S_{k-1}(i-1)) = 0, \\ \mathrm{h}^n(S_i(i-1)) &= \mathrm{h}^n(S_{k-1}(i-1)) = 0. \end{aligned}$$

Applying cohomology to (14) we thus get

$$\mathrm{H}^q\left(S_i \otimes S_i^{\vee}\right) = \mathrm{H}^{q+1}\left(S_i \otimes S_{i-1}^{\vee}\right), \quad \forall\, q \geq 0. \tag{15}$$

Now, consider the sequence obtained after twisting (13) by $S_{i-1}^{\vee}$:

$$0 \to S_i \otimes S_{i-1}^{\vee} \to S_{i-1}^{\vee}(-i+1)^{a_{i-1}} \to S_{i-1} \otimes S_{i-1}^{\vee} \to 0.$$

We claim that $\mathrm{H}^q\left(S_{i-1}^{\vee}(-i+1)\right) = 0$, for all $q \geq 0$. In fact, we saw in the proof of Proposition 3.5 that

$$\mathrm{h}^0\left(S_{i-1}^{\vee}(-i+1)\right) = 0.$$

Also, using (11) with $m = k-2-i$, we obtain

$$\mathrm{h}^{k-i+1}\left(S_{i-1}^{\vee}(-i+1)\right) = \mathrm{h}^{n-k+i-1}\left(S_{i-1}(i-n-2)\right) = \mathrm{h}^{n-1}\left(S_{k-1}(i-n-2)\right) = 0.$$

Finally, it follows also from Lemma 3.2 that

$$\mathrm{h}^n\left(S_{i-1}^{\vee}(-i+1)\right) = \mathrm{h}^0\left(S_{i-1}(i-n-2)\right) = \mathrm{h}^0\left(S_{k-1}(i-n-2)\right) = 0.$$

Therefore, $\mathrm{H}^q\left(S_{i-1}^{\vee}(-i+1)\right) = 0$, for all $q \geq 0$, as claimed, and together with (15), we get

$$\mathrm{H}^q\left(S_{i-1} \otimes S_{i-1}^{\vee}\right) = \mathrm{H}^{q+1}\left(S_i \otimes S_{i-1}^{\vee}\right) = \mathrm{H}^q\left(S_i \otimes S_i^{\vee}\right), \quad \forall\, q \geq 0.$$

Since E is exceptional by hypothesis, we conclude that S_i is also exceptional, for each $i = 1, \dots, k-1$.

We next prove (b). Suppose that $\chi(\operatorname{End} E) = 1$ and S_i is exceptional, for all $i = 1, \dots, k-1$. We have

$$\chi(\operatorname{End} E) = \chi(\operatorname{End} S_1) + a_0\left(a_0 - \chi\left(S_1^{\vee}\right)\right) \Leftrightarrow 1 = 1 + a_0\left(a_0 - \chi\left(S_1^{\vee}\right)\right),$$

that is, $a_0 = \chi\left(S_1^\vee\right)$. This implies that $\mathrm{H}^0\left(E^\vee\right) = 0$. But $\mathrm{H}^k\left(E^\vee\right) = \mathrm{H}^n\left(E^\vee\right) = 0$ and hence $\mathrm{H}^q\left(E^\vee\right) = 0$, for all q. From the sequence

$$0 \to S_1 \otimes E^\vee \to \left(E^\vee\right)^{a_0} \to E \otimes E^\vee \to 0$$

we deduce that $\mathrm{H}^q\left(E \otimes E^\vee\right) \cong \mathrm{H}^{q+1}\left(S_1 \otimes E^\vee\right)$, for $q \geq 0$. Note that the proof of (15) holds in general and does not depend of the hypothesis in (a), so $\mathrm{H}^{q+1}\left(S_1 \otimes E^\vee\right) \cong \mathrm{H}^q\left(S_1 \otimes S_1^\vee\right)$. Then, $\mathrm{H}^q\left(E \otimes E^\vee\right) \cong \mathrm{H}^q\left(S_1 \otimes S_1^\vee\right)$. Thus E is exceptional, for we are supposing S_1 is exceptional. □

It now results that under the hypothesis of Theorem 3.7 the Betti numbers of an exceptional vector bundle E with a linear resolution of type (4) are completely determined. The pair (a_{k-1}, a_k) is a pair of consecutive terms of the sequence (3) (recall Lemma 2.3) and determines all the other exponents in the resolution of E.

Corollary 3.8. *Let E be an exceptional vector bundle on $\mathbb{P}^n$ with linear resolution* (4) *and $\mathrm{H}^{n-k}(E(k-3-n)) = 0$, with $k \geq 2$. Then the Betti numbers of E satisfy the following relations:*

(a) *a_k and a_{k-1} are two consecutive terms, u_s and u_{s+1} respectively, of the sequence $\{u_s\}_{s\geq 0}$ defined by*

$$\begin{cases} u_0 = 0 \\ u_1 = 1 \\ u_{s+1} = (n+1)u_s - u_{s-1}, \end{cases} \tag{16}$$

(b) $a_i = a_{i+1}(n+1) - a_{i+2}\binom{n+2}{2} + \cdots + (-1)^{k-1-i} a_k \binom{n+k-i}{k-i}, \quad i = 0, \ldots, k-2.$

Proof. Suppose that is E is exceptional and $\mathrm{H}^{n-k}(E(k-3-n)) = 0$. S_{k-1} is a Steiner bundle and by Theorem 3.7 it is exceptional so $\chi(\operatorname{End} S_{k-1}) = 1$. Therefore, it follows directly from Theorem 2.5 that the pair (a_k, a_{k-1}) is of the form (u_s, u_{s+1}), for some $s \geq 2$, where $\{u_s\}_{s\geq 0}$ is the sequence

$$\begin{cases} u_0 = 0 \\ u_1 = 1 \\ u_{s+1} = (n+1)u_s - u_{s-1}. \end{cases}$$

In the course of the proof of Proposition 3.5 we saw that

$$a_i = \chi(\left(S_{i+1}^\vee(-i)\right)) = \mathrm{h}^0\left(S_{i+1}^\vee(-i)\right) = \mathrm{h}^n(S_{i+1}(i-n-1)),$$

for $i = 0, \ldots, k-2$. When $i = k-2$, we get

$$a_{k-2} = \mathrm{h}^n(S_{k-1}(k-3-n)).$$

Since $\mathrm{H}^{n-k}(E(-k-3-n)) = \mathrm{H}^{n-1}(S_{k-1}(-k-3-n)) = 0$, Lemma 3.2 (i) allow us to write a_{k-2} in terms of a_{k-1} and a_k:

$$a_{k-2} = a_{k-1}(n+1) - a_k\binom{n+2}{2}.$$

For $1 \leq i \leq k-3$, we apply Lemma 3.2 (ii) to obtain a general formula for a_i:

$$\begin{aligned}
a_i &= \mathrm{h}^n(S_{i+1}(i-n-1)) = a_{i+1}(n+1) - \mathrm{h}^n(S_{i+2}(i-n-1)) \\
&= a_{i+1}(n+1) - a_{i+2}\binom{n+2}{2} + \mathrm{h}^n(S_{i+3}(i-n-1)) = \dots \\
&= a_{i+1}(n+1) - a_{i+2}\binom{n+2}{2} + \dots + (-1)^{k-3-i}a_{k-2}\binom{n+k-2-i}{k-2-i} \\
&\quad +(-1)^{k-2-i}\mathrm{h}^n(S_{k-1}(i-n-1)).
\end{aligned}$$

To compute $\mathrm{h}^n(S_{k-1}(i-n-1))$ we note that $\mathrm{h}^{n-1}(S_{k-1}(i-n-1)) = 0$ (recall (11)) and use Lemma 3.2 (i) again. We thus obtain

$$\begin{aligned}
a_i \quad = \quad & a_{i+1}(n+1) - a_{i+2}\binom{n+2}{2} + \dots + (-1)^{k-3-i}a_{k-2}\binom{n+k-2-i}{k-2-i} + \\
& (-1)^{k-2-i}a_{k-1}\binom{n+k-i-1}{k-i-1} + (-1)^{k-1-i}a_k\binom{n+k-i}{k-i}.
\end{aligned}$$

□

We would like to point out that the previous corollary would still hold under the weaker assumptions of Proposition 3.5. However, we think that the statement becomes more interesting with the hypotheses of Theorem 3.7.

Remark 3.9. Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution (4) and such that $\mathrm{H}^{n-k}(E(k-3-n)) = 0$, with $k \geq 2$. If the Betti numbers of E satisfy (a) and (b) in Corollary 3.8 then E may not be exceptional. Nevertheless, a converse statement of the referred corollary could be as follows:

Let E be a vector bundle on $\mathbb{P}^n$ with linear resolution (4) and $\mathrm{H}^{n-k}(E(k-3-n)) = 0$, with $k \geq 2$. If S_{k-1} is exceptional and the Betti numbers of E satisfy

$$a_i = a_{i+1}(n+1) - a_{i+2}\binom{n+2}{2} + \dots + (-1)^{k-1-i}a_k\binom{n+k-i}{k-i}, \ i = 0, \dots, k-2,$$

then E is exceptional (note that the condition that $(a_k, a_{k-1}) = (u_s, u_{s+1})$, where $\{u_s\}_{s\geq 0}$ is the sequence (16), is automatically satisfied if S_{k-1} is exceptional).

The next example shows that the cohomological vanishing $\mathrm{H}^{n-k}(E(k-3-n)) = 0$ is indeed necessary. We construct a vector bundle of homological dimension 2 whose Betti numbers do not satisfy conditions (a) and (b) in Corollary 3.8, although it is exceptional.

Example 3.10. Set $R = K[x_0, x_1, x_2, x_3, x_4]$. Let A be a homogeneous 3×7 matrix with general linear entries and $I_3(A)$ the ideal generated by the 3×3 minors of A. Hence, by [MR08], Proposition 1.2.16, $R/I_3(A)$ has a minimal free R-resolution of the following type:

$$0 \to R(-7)^{15} \to R(-6)^{70} \to R(-5)^{126} \to R(-4)^{105} \to R(-3)^{35} \to R \to R/I_3(A) \to 0$$

Since $R/I_3(A)$ is an artinian ring then the sheafification of $R/I_3(A)$ is trivial and the corresponding complex in $\mathbb{P}^4$ is as follows:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-7)^{15} \to \mathcal{O}_{\mathbb{P}^4}(-6)^{70} \to \mathcal{O}_{\mathbb{P}^4}(-5)^{126} \to \mathcal{O}_{\mathbb{P}^4}(-4)^{105} \to \mathcal{O}_{\mathbb{P}^4}(-3)^{35} \to \mathcal{O}_{\mathbb{P}^4} \to 0.$$

Twist it by $\mathcal{O}_{\mathbb{P}^4}(5)$ and cut it into short exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{15} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{70} \to S_1 \to 0,$$
$$0 \to S_1 \to \mathcal{O}_{\mathbb{P}^4}^{126} \to E \to 0,$$
$$0 \to E \to \mathcal{O}_{\mathbb{P}^4}(1)^{105} \to F \to 0,$$
$$0 \to F \to \mathcal{O}_{\mathbb{P}^4}(2)^{35} \to \mathcal{O}_{\mathbb{P}^4}(5) \to 0.$$

In particular, the vector bundle E is defined by the linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{15} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{70} \to \mathcal{O}_{\mathbb{P}^4}^{126} \to E \to 0,$$

and $\operatorname{hd} E \leq 2$. Observing that $F^\vee$ is a general Steiner bundle which is exceptional by Theorem 2.5, we infer that F is exceptional.

Using Proposition 3.4 we get $\chi(\operatorname{End} E) = 1$. Let us show that E is exceptional. Consider the sequence

$$0 \to E \otimes F^\vee \to E(-1)^{105} \to E \otimes E^\vee \to 0.$$

Applying the cohomology functor to the sequences

$$0 \to S_1(-1) \to \mathcal{O}_{\mathbb{P}^4}(-1)^{126} \to E(-1) \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{15} \to \mathcal{O}_{\mathbb{P}^4}(-2)^{70} \to S_1(-1) \to 0,$$

we easily deduce that $\mathrm{H}^q(E(-1)) = \mathrm{H}^{q+1}(S_1(-1)) = 0$, for all $q \geq 0$. To compute the cohomology groups of $E \otimes F^\vee$, we look at the cohomology sequence of

$$0 \to E(-5) \to E(-2)^{35} \to E \otimes F^\vee \to 0.$$

To obtain $\mathrm{H}^q(E(-2))$ we proceed similarly as in the computation of $\mathrm{H}^q(E(-1))$ and get $\mathrm{H}^q(E(-2)) = \mathrm{H}^{q+1}(S_1(-2)) = 0$, for every $q \geq 0$. Now, consider the dual sequences

$$0 \to E^\vee \to \mathcal{O}_{\mathbb{P}^4}^{126} \to S_1^\vee \to 0,$$
$$0 \to F^\vee \to \mathcal{O}_{\mathbb{P}^4}(-1)^{105} \to E^\vee \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{O}_{\mathbb{P}^4}(-2)^{35} \to F^\vee \to 0.$$

We have $\mathrm{H}^q(F^\vee) = 0$, $q \neq 3$, and $\mathrm{H}^3(F^\vee) = K$. So $\mathrm{H}^q(E^\vee) \cong \mathrm{H}^{q+1}(F^\vee) = 0$, $q \neq 2$, and $\mathrm{H}^2(E^\vee) \cong \mathrm{H}^3(F^\vee) = K$. This implies that $\mathrm{H}^0(S_1^\vee) = K^{126}$, $\mathrm{H}^1(S_1^\vee) = K$ and $\mathrm{H}^q(S_1^\vee) = 0$, $q \geq 2$. We thus have

$$\mathrm{H}^2(E(-5)) \cong \mathrm{H}^2(E^\vee) = K,$$
$$\mathrm{H}^q(E(-5)) \cong \mathrm{H}^{4-q}(E^\vee) = 0, \ q \neq 2.$$

Finally, this enables us to conclude that

$$\mathrm{H}^q\left(E\otimes E^\vee\right)=\mathrm{H}^{q+1}\left(E\otimes F^\vee\right)=0,\ q\neq 0,$$
$$\mathrm{H}^0\left(E\otimes E^\vee\right)=\mathrm{H}^1\left(E\otimes F^\vee\right)=K,$$

that is, E is exceptional. Note, however, that $\mathrm{H}^{n-k}(E(k-3-n))=\mathrm{H}^2(E(-5))=K$.

Therefore, E is an exceptional bundle such that $\mathrm{H}^2(E(-5))\neq 0$ and its Betti numbers do not satisfy Corollary 3.8. Besides, we note that the associated vector bundle S_1 is not exceptional.

4. Examples

After the results of the previous section we would like to ensure that we can provide examples of exceptional vector bundles of arbitrary homological dimension given by a linear resolution. In this section we are able to deal with the case $k=2$ and we construct examples of vector bundles on the projective spaces of dimension 4 and 5 of various homological dimensions.

Our first example is a family of vector bundles of homological dimension 2 satisfying conditions of Theorem 3.7. In order to construct this family we will use as a tool the following theorem.

Theorem 4.1 ([EH92], Theorem 1)**.** *Suppose $n\geq 2$, $a\geq 1$, and $b\geq (n+3)a/2+1$. For $a\leq 3$, suppose also:*

- *if $a=1$ then $b\geq n+1$;*
- *if $a=2$ then $b\geq 2n+2$;*
- *if $a=3$ then $b\geq 2n+4$ (and $b\geq 9$ for $n=2$).*

Then the kernel $E_{a,b}$ of a general morphism $\mathcal{O}_{\mathbb{P}^n}(1)^b\to\mathcal{O}_{\mathbb{P}^n}(2)^a$ is a locally free sheaf with natural cohomology and generated by its global sections.

Remark 4.2. We note that any pair $(a,b)=(u_s,u_{s+1})$ of consecutive terms of the sequence (16) satisfy the conditions in [EH92].

If $u_s=1$ then $u_{s+1}=n+1$, so obviously $u_{s+1}=n+1\geq u_s=1$. The cases $u_s=2$ and $u_s=3$ never occur.

Consider $b_s\geq 3$ and let us show this by induction on s that $2u_{s+1}\geq (n+3)u_s+2$. If $s=1$ then $2u_2=2n+2\geq n+5=(n+3)u_1+2$, for $n\geq 3$.

Suppose $2u_{t+1}\geq (n+3)u_t+2$, for all $t\leq s-1$. Using the recurrence formula $u_{s+1}=(n+1)u_s-u_{s-1}$ and the induction hypothesis, we get

$$\begin{aligned}2u_{s+1}-(n+3)u_s-2&=2(n+1)u_s-2u_{s-1}-(n+3)u_s-2\\&\geq 2nu_s+(n+3)u_{s-1}+2-2u_{s-1}-(n+3)u_s-2\\&=(n+1)u_{s-1}+(n-3)u_s\geq (n+1)u_{s-1}\geq 0.\end{aligned}$$

Hence, $2u_{s+1}\geq (n+3)u_s+2$.

Now, we are able to prove the following.

Proposition 4.3. *Suppose $n \geq 3$, $a \geq 2$ and $b \geq (n+3)a/2 + 1$. If $a = 1$ suppose also $b \geq n+1$.*

Then there exists an exceptional vector bundle E of homological dimension 2 and linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^a \to \mathcal{O}_{\mathbb{P}^n}(-1)^b \to \mathcal{O}_{\mathbb{P}^n}^c \to E \to 0. \tag{17}$$

In particular, this applies to $(a,b) = (u_s, u_{s+1})$, for some $s \geq 1$, where $\{u_s\}_{s\geq 0}$ is the sequence (16)*, and*

$$c = b(n+1) - a\binom{n+2}{2}.$$

Proof. Let $\phi : \mathcal{O}_{\mathbb{P}^n}(1)^b \to \mathcal{O}_{\mathbb{P}^n}(2)^a$ be a general morhism. Then, by Theorem 4.1, the kernel $F_{b,a}$ of this morphism is a vector bundle with natural cohomology and generated by its global sections. In particular, this implies both that

$$c := \mathrm{h}^0(F_{b,a}) = b(n+1) - a\binom{n+2}{2}, \ \ \mathrm{h}^1(F_{b,a}) = 0,$$

and that there is an epimorphism $\rho : \mathcal{O}_{\mathbb{P}^n}^c \twoheadrightarrow F_{b,a}$. Hence, we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \ker\rho & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}_{\mathbb{P}^n}^c & & & & & & \\
 & & {\scriptstyle\rho}\downarrow & \searrow & & & & & \\
0 & \longrightarrow & F_{b,a} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1)^b & \xrightarrow{\phi} & \mathcal{O}_{\mathbb{P}^n}(2)^a & \longrightarrow & 0. \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Therefore, the vector bundle $E_{b,a} := (\ker\rho)^\vee$ has linear resolution of type (17) and $\mathrm{hd}(E_{b,a}) \leq 2$.

Let us check that $\mathrm{hd}(E_{b,a}) = 2$. If $\mathrm{hd}(E_{b,a}) \leq 1$ then by Lemma 3.1 we would have $\mathrm{H}^{n-2}_*(E_{b,a}) = 0$. However, from the above diagram we see that $\mathrm{H}^{n-2}_*(E_{b,a}) \cong \mathrm{H}^{n-1}_*(F_{b,a}^\vee)$ and we can compute

$$\mathrm{H}^{n-2}(E_{b,a}(-n+1)) \cong \mathrm{H}^{n-1}\left(F_{b,a}^\vee(-n+1)\right) \cong \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)) \neq 0.$$

Now, take an arbitrary pair (a,b) of the form (u_s, u_{s+1}), for some $s \geq 1$, where $\{u_s\}_{s\geq 0}$ is the sequence (16). It follows from Remark 4.2 that $a = u_s$ and $b = u_{s+1}$ satisfy the inequalities $a \geq 2$, $b \geq (n+3)a/2+1$, and $b \geq n+1$ whenever $a = 1$. Then the above construction gives us a vector bundle E_{u_{s+1},u_s} of homological dimension 2, with minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^{u_s} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{u_{s+1}} \to \mathcal{O}_{\mathbb{P}^n}^{u_{s+1}(n+1)-u_s\binom{n+2}{2}} \to E_{u_{s+1},u_s} \to 0.$$

Let us check that E satisfies the hypotheses of Theorem 3.7 (a). The cohomological condition is

$$H^{n-2}\left(E_{u_{s+1},u_s}(-1-n)\right) = H^{n-1}\left(F^\vee_{u_{s+1},u_s}(-1-n)\right) = H^1(F_{u_{s+1},u_s}),$$

and we have seen above that this group vanishes since F_{u_{s+1},u_s} is a globally generated vector bundle with natural cohomology. From Proposition 3.4 we can easily see that $\chi(\operatorname{End} E_{u_{s+1},u_s}) = 1$. Also, $F^\vee_{u_{u_{s+1}},u_s}$ is a general exceptional Steiner bundle. Altogether, by Theorem 3.7 (a), we conclude that E_{u_{s+1},u_s} is exceptional. □

More generally, we are able to construct several examples in the software system Macaulay2 ([GS]).

Example 4.4. In $\mathbb{P}^4$, we consider the pair $(u_1, u_2) = (5, 24)$ of two terms of the sequence (16). The generation of a random 24×5 matrix M produces the following two examples of homological dimensions 2 and 3:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^5 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-1)^{24} \to \mathcal{O}_{\mathbb{P}^4}^{45} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^5 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-2)^{24} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{45} \to \mathcal{O}_{\mathbb{P}^4}^{40} \to E \to 0.$$

Taking the subsequent pair $(u_1, u_2) = (24, 115)$, we construct:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-2)^{24} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-1)^{115} \to \mathcal{O}_{\mathbb{P}^4}^{215} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3)^{24} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}(-2)^{115} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{215} \to \mathcal{O}_{\mathbb{P}^4}^{190} \to E \to 0.$$

In $\mathbb{P}^5$, we proceed in a similar way. The pair $(u_1, u_2) = (6, 35)$ produces the following resolutions:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2)^6 \to \mathcal{O}_{\mathbb{P}^5}(-1)^{35} \to \mathcal{O}_{\mathbb{P}^5}^{84} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^5}(-3)^6 \to \mathcal{O}_{\mathbb{P}^5}(-2)^{35} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{84} \to \mathcal{O}_{\mathbb{P}^5}^{105} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^6 \to \mathcal{O}_{\mathbb{P}^5}(-3)^{35} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{84} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{105} \to \mathcal{O}_{\mathbb{P}^5}^{70} \to E \to 0.$$

The pair $(35, 204)$ enables us to construct:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-2)^{35} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{204} \to \mathcal{O}_{\mathbb{P}^5}^{489} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^5}(-3)^{35} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{204} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{489} \to \mathcal{O}_{\mathbb{P}^5}^{610} \to E \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^{35} \to \mathcal{O}_{\mathbb{P}^5}(-3)^{204} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{489} \to \mathcal{O}_{\mathbb{P}^5}(-1)^{610} \to \mathcal{O}_{\mathbb{P}^5}^{405} \to E \to 0.$$

In each case, one can check with Macaulay2 that $\mathrm{H}^{n-k}(E(k-3-n)) = 0$ and E is exceptional. Note, furthermore, that the Betti numbers satisfy (a) and (b) in Corollary 3.8.

We would like to generalize the construction in the proof of Proposition 4.3 to any homological dimension. Therein, Ellia and Hirshowitz's theorem 4.1 provides us with a vector bundle E of homological dimension 2. If $E^\vee(1)$ is generated by its global sections then there is a surjective morphism $\mathcal{O}_{\mathbb{P}^n}^c \twoheadrightarrow E^\vee(1)$, for some $c \geq 1$, which allows us to produce a new vector bundle of homological dimension 3.

The problem we run into is that we are not able to ensure that the vector bundle $E^\vee(1)$ is generated by its global sections. In fact, $E^\vee(1)$ may not be even 0-regular. According to our notation, this vector bundle corresponds to $S_{k-2}^\vee(1)$ in the examples obtained in 4.4. We observe that $S_{k-2}^\vee(1)$ is always globally generated (but it is not 0-regular). This leads us to propose the following conjecture:

Conjecture 4.5. *Let* $\phi : \mathcal{O}_{\mathbb{P}^n}(1)^b \to \mathcal{O}_{\mathbb{P}^n}(2)^a$ *be a general morphism and* $S := \operatorname{coker} \phi^\vee$. *If* a *and* b *satisfy the hypotheses in Proposition 4.3 there is a diagram*

$$\begin{array}{ccccccccc}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2)^a & \xrightarrow{\phi^\vee} & \mathcal{O}_{\mathbb{P}^n}(-1)^b & & \longrightarrow & & \mathcal{O}_{\mathbb{P}^n}^c \longrightarrow E \longrightarrow 0, \\
 & & & \searrow & & \nearrow & \\
 & & & & S & & \\
 & & & \nearrow & & \searrow & \\
 & & 0 & & & & 0
\end{array}$$

and $E^\vee(1)$ *is generated by its global sections.*

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