Schwarzenberger bundles on smooth projective varieties

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Abstract

We define Schwarzenberger bundles on smooth projective varieties and we introduce the notions of jumping subspaces and jumping pairs of $(F_0, \mathcal{O}_X)$-Steiner bundles. We determine a bound for the dimension of the set of jumping pairs. We classify those Steiner bundles whose set of jumping pairs has maximal dimension by proving that they are Schwarzenberger bundles.

Keywords Projective varieties; Schwarzenberger bundles; Steiner bundles
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Introduction

Steiner vector bundles on projective spaces were first defined by Dolgachev and Kapranov in [DK93] as vector bundles E fitting in an exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^s \to \mathcal{O}_{\mathbb{P}^n}^t \to E \to 0.$$

The authors use them to study logarithmic bundles $E(\mathcal{H}) = \Omega(\log \mathcal{H})$ associated to an arrangement $\mathcal{H}$ of k hyperplanes with normal crossing. They show that logarithmic bundles are rank n Steiner vector bundles on $\mathbb{P}^n$ and get a Torelli type theorem. More precisely, they prove that for $k \geq 2n+3$ the correspondence $\mathcal{H} \to E(\mathcal{H})$ is bijective except when all hyperplanes osculate the same rational normal curve. In this case, $E(\mathcal{H})$ is the Schwarzenberger bundle associated to this curve, as first constructed in [Sch61].

In [Val00], Vallès generalized this result for $k > n+1$. Whereas the main tool in the first paper is the study of the jumping lines of $E(\mathcal{H})$, Vallès focus on a special family of hyperplanes, called unstable hyperplanes. He proves that if a Steiner bundle E has at least $t+2$ unstable hyperplanes in general linear position then all hyperplanes osculate a rational normal curve and E is the Schwarzenberger bundle associated to this curve. Sharing the same idea of unstable hyperplanes, Ancona and Ottaviani show that a Steiner bundle is logarithmic if and only if it contains at least $t+1$ unstable hyperplanes (see [AO01]). Moreover, Vallès sees that this correspondence between Schwarzenberger bundles and rational normal curves is also bijective in the following sense: given a rational normal curve, one can construct the associated Schwarzenberger bundle and reconstruct the curve from its set of unstable hyperplanes.

The generalization of the above correspondence was recently addressed by the first author in [Arr10]. Arrondo introduces the notion of Schwarzenberger bundles on $\mathbb{P}^n$ of arbitrary rank and the ensuing generalization of unstable hyperplanes, which he calls jumping subspaces. A Schwarzenberger bundle will still be a Steiner bundle and in his paper Arrondo studies the problems of when is the latter a Schwarzenberger bundle and the related Torelli-type theorem.

He gets a sharp bound for the dimension of the set of jumping hyperplanes and shows that in the case of maximal dimension all Steiner bundles are Schwarzenberger bundles.

In the end of his work, Arrondo proposes to use the definition of Steiner bundles given in [MRS09] to get a natural definition of Schwarzenberger bundles on other smooth projective varieties. In [AM14], the results in [Arr10] are extended for the Grassmannian variety $\mathbb{G}(k,n)$ and are the main motivation of the present paper. Our goal is to generalize the work in [Arr10] and [AM14] to any smooth projective variety X.

An (F_0,F_1) - Steiner bundle E on X is a vector bundle on X defined by an exact sequence of the form $0 \to F_0^s \to F_1^t \to E \to 0$, where (F_0,F_1) is a strongly exceptional ordered pair of vector bundles on X such that $F_0^\vee \otimes F_1$ is generated by global sections. These bundles were introduced in [Soa08] and a cohomological characterization can be found in [MRS09]. In the above cited papers, Arrondo and Marchesi define Schwarzenberger bundles on $\mathbb{P}^n$ and $\mathbb{G}(k,n)$. They are, respectively, $(\mathcal{O}_{\mathbb{P}^n}(-1),\mathcal{O}_{\mathbb{P}^n})$ and $(\mathcal{U},\mathcal{O}_{\mathbb{G}(k,n)})$-Steiner bundles obtained from a triplet (Z,L,M), where Z is any projective variety, and L, M are globally generated vector bundles on Z. In order to generalize these concepts in a natural way we will restrict our study to $(F_0,\mathcal{O}_X)$-Steiner bundles. Denoting $f_0 = \mathrm{rk}(F_0)$, a Schwarzenberger on X will be a Steiner bundle obtained from the data (Z,ψ,L), where Z is a projective variety provided with a non-degenerate linearly normal morphism $\psi : Z \to G(f_0,H^0(F_0^\vee))$, and L is a globally generated vector bundle on Z. When X is the Grassmannian variety $\mathbb{G}(k,n)$ (which includes the projective space case), we get a Schwarzenberger bundle according to Arrondo and Marchesi, when $\psi : Z \to \mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. Moreover, this definition will allow us to generalize the notion of an (a,b)-jumping pair for an $(F_0,\mathcal{O}_X)$-Steiner bundle E and prove a Torelli-type theorem when $a = 1$ and $b = f_0$. That is we will show that, in the case when the set $\tilde{J}(E)$ of $(1,f_0)$-jumping pairs of E (endowed with a natural structure of a projective variety) has maximal dimension, E is a (Z,ψ,L)-Schwarzenberger bundle and the $Z = \tilde{J}(E)$.

Next we outline the structure of the paper. In Section 1 we recall the definition of Steiner bundles on smooth projective varieties and their basic properties. In particular, we give an equivalent definition in terms of linear algebra and get a low bound for the rank of an $(F_0,\mathcal{O}_X)$-Steiner bundle.

In Section 2 we recall the construction of Schwarzenberger bundles on the Grassmann variety and define Schwarzenberger bundles on smooth projective varieties (Definition 2.1).

In Section 3 we introduce the notions of a jumping subspace and of a jumping pair of a Steiner bundle on X (Definition 3.1) and endow the set of all jumping pairs with the structure of a projective variety. We furthermore give a lower bound for its dimension.

In Section 4 we obtain an upper bound for the dimension of the jumping variety by studying its tangent space at a fixed jumping pair (see Theorem 4.1).

In Section 5 we provide a complete classification of Steiner bundles whose jumping locus has maximal dimension. In particular, we show that they all are Schwarzenberger bundles (Theorem 5.1).

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1 Steiner bundles on smooth projective varieties

In this section we recall the definition of Steiner bundles on smooth projective varieties introduced in [MRS09] and we study some of their properties needed in the sequel.

Let us first fix some notation.

Notation 1.1. We will always work over a fixed algebraically closed field k of characteristic zero and X will always denote a smooth projective variety over k.

The projective space $\mathbb{P}(V)$ will be the set of hyperplanes of a vector space V over k or, equivalently, the set of lines in the dual vector space of V, denoted by V^*.

We will write $\mathbb{G}(r-1, \mathbb{P}(V))$ for the Grassmann variety of $(r-1)$-linear subspaces of the projective space $\mathbb{P}(V)$. This is equivalent to consider the set $G(r, V^*)$ of r-dimensional subspaces of the vector space V^*.

The dual of a coherent sheaf E on X will be denoted by $E^\vee$. If E is a vector bundle on X then, for each $x \in X$, E_x is the fibre over x and $h^i(E)$ denotes the dimension of $H^i(E)$.

In order to define Steiner bundles on a smooth projective variety X we will need the following definition.

Definition 1.2. Let X be a smooth projective variety. A coherent sheaf E on X is *exceptional* if

$$\mathrm{Hom}(E,E) \simeq k,$$
$$\mathrm{Ext}^i(E,E) = 0, \text{ for all } i \geq 1.$$

An ordered pair (E, F) of coherent sheaves on X is called an *exceptional pair* if both E and F are exceptional and

$$\mathrm{Ext}^p(F,E) = 0, \text{ for all } p \geq 0.$$

If, in addition,

$$\mathrm{Ext}^p(E,F) = 0 \text{ for all } p \neq 0,$$

we say that (E, F) is a *strongly exceptional pair*.

Definition 1.3. Let X be a smooth projective variety. An (F_0, F_1)-*Steiner bundle* E on X is a vector bundle on X defined by an exact sequence of the form

$$0 \to S \otimes F_0 \to T \otimes F_1 \to E \to 0,$$

where S and T are vector spaces over k of dimensions s and t, respectively, and (F_0, F_1) is an ordered pair of vector bundles on X satisfying the two following conditions:

(i) (F_0, F_1) is strongly exceptional;

(ii) $F_0^\vee \otimes F_1$ is generated by global sections.

Examples 1.4.

(a) A Steiner bundle, as defined by Dolgachev and Kapranov in [DK93], is an $(\mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n})$-Steiner bundle in the sense of Definition 1.3. More generally, vector bundles E with a resolution of type

$$0 \to \mathcal{O}_{\mathbb{P}^n}(a)^s \to \mathcal{O}_{\mathbb{P}^n}(b)^t \to E \to 0,$$

where $1 \leq b - a \leq n$, are $(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b))$-Steiner bundles on $\mathbb{P}^n$ (see [MRS09]).

(b) Consider the smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$, $n \geq 2$, and let Σ_* denote the Spinor bundle Σ on Q_n if n is odd, and one of the Spinor bundles Σ_+ or Σ_- on Q_n if n is even. The vector bundle E on Q_n defined by an exact sequence of the form

$$0 \to \mathcal{O}_{Q_n}(a)^s \to \Sigma_*(n-1)^t \to E \to 0,$$

for any $0 \leq a \leq n-1$, is an $(\mathcal{O}_{Q_n}(a), \Sigma_*(n-1))$-Steiner bundle (see [MRS09]).

(c) Any exact sequence of vector bundles on the Grassmann variety $\mathbb{G} := \mathbb{G}(r-1, \mathbb{P}(V))$ of the form

$$0 \to \mathcal{U}^s \to \mathcal{O}_{\mathbb{G}}^t \to E \to 0,$$

where $\mathcal{U}$ denotes the rank r universal subbundle of $\mathbb{G}$, defines a $(\mathcal{U}, \mathcal{O}_{\mathbb{G}})$-Steiner bundle E on $\mathbb{G}$. These bundles were studied by Arrondo and Marchesi in [AM14].

(d) Let $X = \widetilde{\mathbb{P}^2}$ be the blow-up of $\mathbb{P}^2$ at three points p_1, p_2 and p_3. Let $K_X = -3L + E_1 + E_2 + E_3$ denote the canonical divisor, where L is the divisor corresponding to a line not passing through any of the three points, and E_i is the exceptional divisor of the blow-up at the point p_i, $i = 1, 2, 3$. Take $F_0 = -2L + E_1 + E_2 + E_3$ and $F_1 = \mathcal{O}_X$.

Observe that $H^0(F_0^\vee)$ is globally generated and is the set of conics that passes through three points. In particular, $h^0(F_0^\vee) = 3 = \dim X + 1$.

Let us now prove that the pair of vector bundles (F_0, F_1) is strongly exceptional. Since both F_0 and F_1 are line bundles on a projective variety, the fact that $\mathrm{Hom}(F_0, F_0) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}$ and $\mathrm{Ext}^i(F_0, F_0) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_X) = 0$, $i = 1, 2$, is straightforward.

Using Riemann-Roch formula we obtain $\chi(F_0^\vee) = 3$ and thus $h^1(F_0^\vee) = h^2(F_0^\vee)$. Since $H^2(F_0^\vee) = H^0(K_X + F_0)^* = H^0(-5L + 2E_1 + 2E_2 + 2E_3)^* = 0$ we get that $\mathrm{Ext}^i(F_0, \mathcal{O}_X) = H^i(F_0^\vee) = 0$, for $i = 1, 2$.

From the fact that $H^0(F_0^\vee) \neq 0$ and $\mathrm{Hom}(F_0, F_0) = H^0(F_0 \otimes F_0^\vee) \neq 0$ it follows that $\mathrm{Hom}(\mathcal{O}_X, F_0)$ must be trivial. Furthermore, $\mathrm{Ext}^2(\mathcal{O}_X, F_0) = H^2(F_0) = H^0(-F_0 + K_X)^* = H^0(-L)^* = 0$. Then, it also holds $\mathrm{Ext}^1(\mathcal{O}_X, F_0) = H^1(F_0) = 0$, for we have $\chi(F_0) = 0$.

So, we have just proved that any vector bundle E fitting in a sequence of type

$$0 \to F_0^s \to \mathcal{O}_X^t \to E \to 0$$

is an $(F_0, \mathcal{O}_X)$-Steiner bundle on the blow-up X.

The following proposition gives a characterization of (F_0, F_1)-Steiner bundles on a smooth projective variety X by means of linear algebra (recall also Lemma 1.2 in [Arr10] or Lemma 1.7 in [AM14]). This interpretation will play an essential role for studying Schwarzenberger bundles on X.

Proposition 1.5. *To give an (F_0, F_1)-Steiner bundle on a smooth projective variety X*

$$0 \to S \otimes F_0 \to T \otimes F_1 \to E \to 0,$$

is equivalent to give a linear map $\varphi : T^* \to S^* \otimes H^0(F_0^\vee \otimes F_1)$ such that, for each $x \in X$, the induced linear map

$$\widetilde{\varphi}_x : T^* \otimes (F_1)_x^* \to S^* \otimes (F_0^\vee)_x$$

is surjective.

Proof. Dualizing the sequence defining the Steiner bundle E we see that to give a map $S \otimes F_0 \to T \otimes F_1$ is the same as to give a map $\widetilde{\varphi} : T^* \otimes F_1^\vee \to S^* \otimes F_0^\vee$.

Twisting by F_1, taking cohomology and using condition (ii) of Definition 1.3, this is clearly equivalent to a linear map $\varphi : T^* \to S^* \otimes H^0(F_0^\vee \otimes F_1)$ with fibers $\varphi_x : T^* \to S^* \otimes H^0((F_0^\vee)_x \otimes (F_1)_x) \cong S^* \otimes (F_0^\vee)_x \otimes (F_1)_x$. Hence, φ_x induces a linear map $\widetilde{\varphi}_x : T^* \otimes (F_1)_x^* \to S^* \otimes (F_0^\vee)_x$ and, moreover, the map $S \otimes F_0 \to T \otimes F_1$ is injective if and only if $\widetilde{\varphi}_x$ is surjective for each $x \in X$.

□

In what follows, φ will always denote the linear map associated to an (F_0, F_1)-Steiner bundle introduced in Proposition 1.5.

Lemma 1.6. *Let E be an (F_0, F_1)-Steiner bundle on a smooth projective variety X. Then the following properties are equivalent:*

(i) *φ is injective.*

(ii) *$H^0(E^\vee \otimes F_1) = 0$.*

(iii) *E cannot split as $E_K \oplus (K^* \otimes F_1)$, where $0 \neq K \subset \ker \varphi \subset T^*$ is a vector space.*

Moreover, if φ is not injective then E_K is the (F_0, F_1)-Steiner bundle corresponding to the map $T^*/K \to S^* \otimes H^0(F_0^\vee \otimes F_1)$. In particular, when $K = \ker \varphi$, there is a splitting $E = E_{\ker \varphi} \oplus ((\ker \varphi)^* \otimes F_1)$, $E_{\ker \varphi}$ is the (F_0, F_1)-Steiner bundle corresponding to the inclusion $\operatorname{Im} \varphi \hookrightarrow S^* \otimes H^0(F_0^\vee \otimes F_1)$ and $H^0(E_{\ker \varphi}^\vee \otimes F_1) = 0$.

Proof. To see that (i) is equivalent to (ii), it is enough to observe that, after dualizing and twisting by F_1 the exact sequence defining E, we get a short exact sequence

$$0 \to E^\vee \otimes F_1 \to (F_1^\vee)^t \otimes F_1 \to (F_0^\vee)^s \otimes F_1 \to 0.$$

So, taking cohomology and using the fact that (F_0, F_1) is an exceptional pair, we see that φ is injective if and only if $H^0(E^\vee \otimes F_1) = 0$.

Now, if E splits as $E_K \oplus (K \otimes F_1)$, for some vector space $K \subset \ker \varphi \subset T^*$, then there is a non-trivial morphism $E \to F_1$, i.e. $H^0(E^\vee \otimes F_1) \neq 0$. This proves that (i) implies (iii).

Reciprocally, suppose φ is not injective and let $0 \neq \tilde{t} \in \ker \varphi \subset T^*$. There is a commutative triangle of linear maps

$$\begin{tikzcd} T^* \arrow[r, "\varphi"] \arrow[d] & S^* \otimes H^0(F_0^\vee \otimes F_1) \\ T^*/\tilde{t} \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

which, by Proposition 1.5, induces a commutative triangle

$$\begin{tikzcd} T^* \otimes (F_1)_x^* \arrow[r, two heads, "\widetilde{\varphi}_x"] \arrow[d] & S^* \otimes (F_0^\vee)_x \\ T^*/\tilde{t} \otimes (F_1)_x^* \arrow[ur, "\widetilde{\varphi'}_x"'] & \end{tikzcd}$$

In particular, we see that φ' is a linear map such that φ'_x induces a surjective linear map $\widetilde{\varphi'}_x : T^* \otimes (F_1)^*_x \to S^* \otimes (F_0^\vee)_x$. Therefore, φ' defines an (F_0, F_1)-Steiner bundle $E_{\tilde{t}}$ and we have a commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
0 \arrow[r] & S \otimes F_0 \arrow[r] \arrow[d, equal] & \left(T^*/\tilde{t}\right)^* \otimes F_1 \arrow[r] \arrow[d] & E_{\tilde{t}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \otimes F_0 \arrow[r] & T \otimes F_1 \arrow[r] \arrow[d] & E \arrow[r] & 0 \\
& & \langle \tilde{t} \rangle^* \otimes F_1 \cong F_1 \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

From the snake's lemma we immediately deduce that the morphism $E_{\tilde{t}} \to E$ is injective and that its cokernel is isomorphic to F_1. Hence, the diagram above can be completed as follows:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S \otimes F_0 \arrow[r] \arrow[d, equal] & \left(T^*/\tilde{t}\right)^* \otimes F_1 \arrow[r] \arrow[d] & E_{\tilde{t}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S \otimes F_0 \arrow[r] & T \otimes F_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
& & \langle \tilde{t} \rangle^* \otimes F_1 \cong F_1 \arrow[r, equal] \arrow[d] & F_1 \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Now, applying the functor $\mathrm{Hom}(F_1, -)$ to the exact sequence that defines E (the middle row in the diagram), it follows that $\mathrm{Hom}(F_1, E) \cong k^t$. Applying the same functor to the first row and the right column in the diagram, we get that $\mathrm{Ext}^1(F_1, E_{\tilde{t}}) = 0$. Thus, E splits as $E_K \oplus (K^* \otimes F_1)$, where by construction $K = \langle \tilde{t} \rangle^* \subset \ker \varphi \subset T^*$.

The last statements follow directly from Proposition 1.5 and the equivalences just proved. $\square$

The previous lemma motivates the following definition.

Definition 1.7. An (F_0, F_1)-Steiner E is called *reduced* if one of the properties in Lemma 1.6 hold. The (F_0, F_1)-Steiner bundle $E_0 := E_{\ker \varphi}$ is called the *reduced summand* of E. In particular, E is reduced if and only if it coincides with its reduced summand.

Examples 1.8.

(a) Any $(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b))$-Steiner bundle on $\mathbb{P}^n$ of rank n is reduced. If this was not the case, the previous lemma would imply that there would exist an $(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b))$-Steiner bundle $E_{\ker\varphi}$ of rank less than n, contradicting Proposition 1.11 below (or Proposition 3.9 in [DK93]).

(b) Let $s \leq k+1$. Then, the following is an exact sequence on the Grassmannian variety $\mathbb{G} := \mathbb{G}(k, \mathbb{P}(V))$, with V a vector space of dimension $n+1$:

$$0 \to \mathcal{U}^s \to \mathcal{O}_{\mathbb{G}}^{s(n+1)+r} \to Q^s \oplus \mathcal{O}_{\mathbb{G}}^{\alpha} \to 0,$$

with $r \geq 0$, and Q is the quotient bundle on $\mathbb{G}$. If $r > 0$ then $Q^s \oplus \mathcal{O}_{\mathbb{G}}^r$ is a $(\mathcal{U}, \mathcal{O}_{\mathbb{G}})$-Steiner bundle on $\mathbb{G}$ that is not reduced (see [AM14], Theorem 2.13).

From now on, and in order to generalize the results in [AM14], we will restrict our study to $(F_0, \mathcal{O}_X)$-Steiner bundles on a smooth projective variety X.

Given an $(F_0, \mathcal{O}_X)$-Steiner bundle, $F_0^\vee$ is a globally generated vector bundle on X and hence we have an exact sequence of vector bundles on X:

$$0 \longrightarrow F_0 \longrightarrow H^0(F_0^\vee)^* \otimes \mathcal{O}_X \longrightarrow Q \longrightarrow 0. \tag{1}$$

When φ is surjective we can determine all $(F_0, \mathcal{O}_X)$-Steiner bundles.

Proposition 1.9. *Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X. Then, φ is surjective if and only if $E \cong (S \otimes Q) \oplus \mathcal{O}_X^p$, for some $p \geq 0$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & E^\vee \arrow[r] \arrow[d] & T^* \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\varphi \otimes id_X"] & S^* \otimes F_0^\vee \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & S^* \otimes Q^\vee \arrow[r] & S^* \otimes H^0(F_0^\vee) \otimes \mathcal{O}_X \arrow[r] & S^* \otimes F_0^\vee \arrow[r] & 0
\end{tikzcd}$$

Suppose φ is surjective. If E is reduced then φ is bijective and it follows from the above diagram that $E \cong S \otimes Q$. If E is not reduced then there is a splitting $E = E_0 \oplus ((\ker\varphi)^* \otimes \mathcal{O}_X)$, where E_0 is the reduced summand of E and the (F_0, F_1)-Steiner bundle corresponding to the bijection $\operatorname{Im}\varphi \xrightarrow{\sim} S^* \otimes H^0(F_0^\vee)$ (recall Lemma 1.6). Hence, $E_0 = S \otimes Q$ and $E = (S \otimes Q) \oplus \mathcal{O}_X^p$, with $p = \dim(\ker\varphi) = t - sh^0(F_0^\vee)$.

Conversely, suppose that $E \cong (S \otimes Q) \oplus \mathcal{O}_X^p$, for some $p \geq 0$. If $p = 0$ then $E \cong S \otimes Q$ and hence $\varphi \otimes id_X$ is bijective. In particular, φ is bijective and thus surjective. If $p > 0$ then E is not reduced, $S \otimes Q$ is its reduced summand and $p = \dim(\ker\varphi)$. In particular, we have a diagram

$$\begin{tikzcd}
T^* \arrow[r, "\varphi"] \arrow[d, "\pi"] & S^* \otimes H^0(F_0^\vee) \\
T^*/(\ker\varphi) \arrow[ur, "\varphi'"'] &
\end{tikzcd},$$

with π and φ' surjective maps. Then φ is also surjective. $\square$

Definition 1.10. Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X. E is a *trivial* $(F_0, \mathcal{O}_X)$*-Steiner bundle (TSB)* if φ is surjective, i.e. if $E \cong (S \otimes Q) \oplus \mathcal{O}_X^p$, for some $p \geq 0$.

Next proposition determines a low bound for the rank of an $(F_0, \mathcal{O}_X)$-Steiner bundle which is not TSB, and thus a necessary condition for its existence.

Theorem 1.11. *Let E be a non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X such that each Chern class of $S \otimes Q$ is non-zero, where Q is as defined in (1). Then*

$$\operatorname{rk}(E) \geq \dim X.$$

Proof. Suppose E is reduced and consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E^\vee \arrow[r] \arrow[d] & T^* \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\varphi \otimes id_X"] & S^* \otimes F_0^\vee \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & S^* \otimes Q^\vee \arrow[r] \arrow[d] & S^* \otimes H^0(F_0^\vee) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & S^* \otimes F_0^\vee \arrow[r] & 0 \\
 & \frac{S^* \otimes H^0(F_0^\vee)}{T^*} \otimes \mathcal{O}_X \arrow[r, "\simeq"] \arrow[d] & \frac{S^* \otimes H^0(F_0^\vee)}{T^*} \otimes \mathcal{O}_X \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then $\operatorname{rk}(E) \leq s \cdot \operatorname{rk}(Q)$, with $\operatorname{rk}(E) = s \cdot \operatorname{rk}(Q)$ if and only if $E \cong S \otimes Q$. Since E is non-TSB then we must have $\operatorname{rk}(E) < s \cdot \operatorname{rk}(Q)$.

Now, let α be the vector bundle morphism defined by

$$S \otimes F_0 \xrightarrow{\alpha} T \otimes \mathcal{O}_X \dashrightarrow E.$$

Let $f_0 = \operatorname{rk}(F_0)$ and consider the *degeneracy locus* of α,

$$D_\alpha = \{x \in X \mid \operatorname{rk}(\alpha_x) \leq s f_0 - 1\},$$

i.e. the set of points x of X such that the rank of the morphism α_x is not maximal. We know that E is a Steiner bundle if and only if α is injective in each fiber, that is, if and only if $D_\alpha = \emptyset$.

By Porteous' formula we have that the expected codimension of the degeneracy locus is equal to

$$t - s f_0 + 1 = \operatorname{rk}(E) + 1.$$

Hence, it is clear that when $\operatorname{rk}(E) + 1 > \dim X$, or equivalently, when $\operatorname{rk}(E) \geq \dim X$, we can ensure that α is injective.

Otherwise, when $\operatorname{rk}(E) + 1 \leq \dim X$, i.e. $\operatorname{rk}(E) < \dim X$, the degeneracy locus will be empty if and only if its fundamental class, given by the Chern class $c_{r+1}(S \otimes Q)$, is zero. From our hypothesis on the Chern classes of $S \otimes Q$, this holds if and only if $r + 1 > \operatorname{rk}(S \otimes Q)$, i.e $\operatorname{rk}(E) \geq s \cdot \operatorname{rk}(Q)$, leading us to a contradiction.

Therefore, $\operatorname{rk}(E) \geq \dim X$.

If E is not reduced then there is a splitting $E = E_K \oplus (K^* \otimes \mathcal{O}_X)$, where E_K is a reduced $(F_0, \mathcal{O}_X)$-Steiner bundle. So by the previous argument, $\operatorname{rk}(E) \geq \operatorname{rk}(E_K) \geq \dim X$.

$\square$

Remark 1.12. Let us analyze in more detail the reason we required all Chern classes of $S \otimes \mathcal{Q}$ to be non-zero.
Suppose, on the contrary, that there exists a zero Chern class, i.e. $c_i(S \otimes Q) = 0$ for a fixed i. Observe that it is always possible to obtain, by Porteous formula as before, a short exact sequence of the form

$$0 \longrightarrow S \otimes F_0 \xrightarrow{\alpha} T \otimes \mathcal{O}_X \longrightarrow E \longrightarrow 0$$

with $\mathrm{rk}(E) = \dim X$. Since E is globally generated and has $c_i(E) = c_i(S \otimes \mathcal{Q}) = 0$, we get a short exact sequence of bundles (see Lemma 4 of [Tan76])

$$0 \longrightarrow \mathcal{O}_X^{\dim X - i+1} \longrightarrow E \longrightarrow F \longrightarrow 0,$$

with $\mathrm{rk}(F) < \dim X$.
Hence, without our hypotheses on the Chern classes in Theorem 1.11 we would be looking for vector bundles whose rank is lower than the dimension of the base variety. This represents a different and challenging problem in the study of vector bundles and in this paper we will not deal with it.

Remark 1.13. It follows from Theorem 1.11 and its proof that given an $(F_0, \mathcal{O}_X)$-Steiner bundle E on X such that each Chern class of $S \otimes Q$ is non-zero and $\mathrm{rk}(E) < \dim X$ one must have $E \cong (S \otimes Q) \oplus \mathcal{O}_X^p$, for some $p \geq 0$ and $\dim X > \mathrm{rk}(E) \geq s\mathrm{rk}(Q)$.

2 Generalized Schwarzenberger on smooth projective varieties

Our goal in this section is to generalize Schwarzenberger bundles on the projective space and on the Grassmann variety $\mathbb{G}(k, n)$, as defined in [Arr10] and [AM14], respectively, to any smooth projective variety X.

We first recall the definition of Schwarzenberger bundles on $\mathbb{G}(k, n)$, following [AM14]. Let L, M be two globally generated vector bundles over a projective variety Z, with $h^0(M) = n+1$ and the identification $\mathbb{P}^n = \mathbb{P}(H^0(M)^*)$. Consider the composition

$$H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L) \otimes H^0(M) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{G}}.$$

We want this composition to be injective, that is we want it be injective in each fiber. This is equivalent to fixing $k+1$ independent global sections $\{\sigma_1, \dots, \sigma_{k+1}\}$ in $H^0(M)$ in correspondence to the point $\Gamma = [< \sigma_1, \dots, \sigma_{k+1} >] \in \mathbb{G}(k, n)$ and requiring the injectivity of the following composition

$$H^0(L) \otimes < \sigma_1, \dots, \sigma_{k+1} > \longrightarrow H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M),$$

given by multiplication with the global section subspace $< \sigma_1, \dots, \sigma_{k+1} >$.

If the injectivity holds for each point of the Grassmannian then a *Schwarzenberger bundle* $F = F(Z, L, M)$ on $\mathbb{G} = \mathbb{G}(k, n)$ is the $(\mathcal{U}, \mathcal{O}_{\mathbb{G}})$-Steiner bundle defined by the resolution

$$0 \longrightarrow H^0(L) \otimes \mathcal{U} \longrightarrow H^0(L \otimes M) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow F \longrightarrow 0.$$

The previous construction motivates the following definition of Schwarzenberger bundles on a smooth projective variety X. Let F_0 be a rank f_0 bundle on X with $F_0^\vee$ generated by global sections and L a globally generated locally free sheaf on a projective variety Z. Take a non-degenerate linearly normal morphism $\psi : Z \to G(f_0, H^0(F_0^\vee))$ and consider the composition

$$0 \to H^0(L)\otimes F_0 \to H^0(L)\otimes H^0(F_0^\vee)^*\otimes\mathcal{O}_X \to H^0(L)\otimes H^0(\psi^*\mathcal{U}^\vee)\otimes\mathcal{O}_X \to H^0(L\otimes\psi^*\mathcal{U}^\vee)\otimes\mathcal{O}_X,$$

where $\mathcal{U}$ denotes the universal subbundle on $G := G(f_0, H^0(F_0^\vee))$.

The first map is given by the monomorphism $F_0 \hookrightarrow H^0(F_0^\vee)^* \otimes \mathcal{O}_X$ (recall that $F_0^\vee$ is generated by global sections and thus there is an epimorphism $H^0(F_0^\vee) \otimes \mathcal{O}_X \twoheadrightarrow F_0^\vee$). The second map is just given by the fact that $H^0(F_0^\vee)^* \cong H^0(\mathcal{U}^\vee) \cong H^0(\psi^*\mathcal{U}^\vee)$. The last map is the one induced by the natural morphism $H^0(L) \otimes H^0(\psi^*\mathcal{U}^\vee) \to H^0(L \otimes \psi^*\mathcal{U}^\vee)$.

Let us show that this composition is injective, i.e. that

$$\eta : H^0(L) \otimes F_0 \to H^0(L \otimes \psi^*\mathcal{U}^\vee) \otimes \mathcal{O}_X$$

is injective on each fiber. Given any $x \in X$, the composition $(F_0)_x \to H^0(\psi^*\mathcal{U}^\vee)$ of the first two maps (on the second factor) is obviously injective. Observing that to give a morphism $(F_0)_x \hookrightarrow H^0(\psi^*\mathcal{U}^\vee)$ is the same as to give a map $(F_0)_x \otimes \mathcal{O}_Z \hookrightarrow \psi^*\mathcal{U}^\vee$, we deduce that $L \otimes (F_0)_x \hookrightarrow L \otimes \psi^*\mathcal{U}^\vee$ is still injective. Finally, applying cohomology, we conclude that $\eta_x : H^0(L) \otimes (F_0)_x \hookrightarrow H^0(L \otimes \psi^*\mathcal{U}^\vee)$ is injective.

Therefore, we have just constructed an $(F_0, \mathcal{O}_X)$-Steiner bundle on X defined by

$$0 \to H^0(L) \otimes F_0 \to H^0(L \otimes \psi^*\mathcal{U}^\vee) \otimes \mathcal{O}_X \to E \to 0.$$

Furthermore, observe that, under the identification $H^0(F_0^\vee)^* \cong H^0(\psi^*\mathcal{U}^\vee)$, the associated linear map $\varphi : H^0(L\otimes\psi^*\mathcal{U}^\vee)^* \to H^0(L)^* \otimes H^0(F_0^\vee)$ of E is nothing but the dual of the multiplication map $H^0(L) \otimes H^0(\psi^*\mathcal{U}^\vee) \to H^0(L \otimes \psi^*\mathcal{U}^\vee)$.

This construction allows us to generalize the notion of a Schwarzenberger bundle to any smooth projective variety.

Definition 2.1. Let X be a smooth projective variety and F_0 a rank f_0 bundle on X such that $F_0^\vee$ is generated by global sections. Let Z be a projective variety together with a non-degenerate linearly normal morphism $\psi : Z \to G(f_0, H^0(F_0^\vee))$, and L a globally generated vector bundle on Z. A (Z, ψ, L)-*Schwarzenberger bundle* on X is an $(F_0, \mathcal{O}_X)$-Steiner bundle E defined by the short exact sequence

$$0 \to H^0(L) \otimes F_0 \to H^0(L \otimes \psi^*\mathcal{U}^\vee) \otimes \mathcal{O}_X \to E \to 0$$

constructed above.

Remark 2.2. Observe that a Schwarzenberger bundle $F = F(Y, L, M)$ on $\mathbb{G}(k, n)$ is a (Z, ψ, L)-Schwarzenberger bundle in the sense of Definition 2.1, with $Z = Y$ and $\psi : Y \to \mathbb{P}^n = \mathbb{P}(H^0(M)^*)$.

3 Jumping pairs of Steiner bundles

We would like to know when an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X is a (Z, ψ, L)-Schwarzenberger bundle on X. In order to answer to this question we will look for a distinctive feature of the Schwarzenberger bundles. More precisely, we will show that a (Z, ψ, L)-Schwarzenberger bundle on X associates a special subspace of $H^0(F_0^\vee)$ to each point $z \in Z$.

If E is a (ψ, Z, L)-Schwarzenberger bundle on X, we have

$$S = H^0(L), \quad T = H^0(L \otimes \psi^* U^\vee),$$

and we already observed that the associated map $\varphi : H^0(L \otimes \psi^* U^\vee)^* \to H^0(L)^* \otimes H^0(F_0^\vee)$ of E is the dual of the multiplication map. Denote $\mathrm{rk}(L) = a$.

For each $z \in Z$, the surjective morphisms $H^0(\psi^* U^\vee) \twoheadrightarrow (\psi^* U^\vee)_z$ and $H^0(L) \twoheadrightarrow L_z$ induce, respectively, an f_0-dimensional subspace $(\psi^* U)_z \subset H^0(\psi^* U^\vee)^* \cong H^0(F_0^\vee)$ and a subspace $L_z^* \subset H^0(L)^*$ of dimension a. Since φ maps $H^0(L_z \otimes (\psi^* U^\vee)_z)^*$ isomorphically into $H^0(L_z)^* \otimes H^0((\psi^* U^\vee)_z)^* \cong L_z^* \otimes (\psi^* U)_z$, then each point $z \in Z$ yields a pair of subspaces $(L_z^*, (\psi^* U)_z)$ such that $L_z \otimes (\psi^* U)_z \in \mathrm{Im}\, \varphi$.

This property of the Schwarzenberger bundles motivates the following definitions.

Definition 3.1. Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X. An *(a,b)-jumping subspace of E* is a b-dimensional subspace $B \subset H^0(F_0^\vee)$ for which there exists an a-dimensional subspace $A \subset S^*$ such that $A \otimes B$ is in the image T_0^* of $\varphi : T^* \to S^* \otimes H^0(F_0^\vee)$. Such a pair (A, B) is called an *(a,b)-jumping pair of E*.

We will write $J_{a,b}(E)$ and $\tilde{J}_{a,b}(E)$ to denote, respectively, the set of (a,b)-jumping subspaces and the set of (a,b)-jumping pairs of E. Moreover, we will write $\Sigma_{a,b}(E)$ to denote the set of a-dimensional subspaces $A \subset S^*$ for which there exists a b-dimensional subspace $B \subset H^0(F_0^\vee)$ such that (A, B) is an (a,b)-jumping pair of E.

It turns out that the set of jumping pairs has a geometric interpretation similar to the one obtained in Lemma 2.4 in [Arr10] and that endows $\tilde{J}_{a,b}(E)$ with a natural structure of a projective variety. Consider the natural generalized Segre embedding

$$\nu : G(a, S^*) \times G(b, H^0(F_0^\vee)) \to G(ab, S^* \otimes H^0(F_0^\vee))$$

given by the tensor product of subspaces. Then, we can state the following result (we will omit the proof for it is essentially the same as that of Lemma 2.4 in [Arr10]).

Lemma 3.2. *Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X and let $T_0^* = \mathrm{im}\, \varphi$. Then:*

(i) the set $\tilde{J}_{a,b}(E)$ of jumping pairs of E is the intersection of the image of ν with the subset $G(ab, T_0^*) \subset G(ab, S^* \otimes H^0(F_0^\vee))$, i.e.

$$\tilde{J}_{a,b}(E) = \mathrm{Im}\, \nu \cap G(ab, T_0^*).$$

(ii) If π_1 and π_2 are the respective projections from $\tilde{J}_{a,b}(E)$ to $G(a, S^*)$ and $G(b, H^0(F_0^\vee))$, then $\Sigma_{a,b}(E) = \pi_1(\tilde{J}_{a,b}(E))$ and $J_{a,b}(E) = \pi_2(\tilde{J}_{a,b}(E))$.

(iii) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Q}$ be the universal quotient bundles of respective ranks a, b and ab of $G(a, S^*)$, $G(b, H^0(F_0^\vee))$ and $G(ab, T_0^*)$. Assume that the natural maps

$$\alpha : H^0(G(a, S^*), \mathcal{A}) \to H^0(\tilde{J}_{a,b}(E), \pi_1^* \mathcal{A})$$
$$\beta : H^0(G(b, H^0(F_0^\vee)), \mathcal{B}) \to H^0(\tilde{J}_{a,b}(E), \pi_2^* \mathcal{B})$$
$$\gamma : H^0(G(ab, T_0^*), \mathcal{Q}) \to H^0(\tilde{J}_{a,b}(E), \mathcal{Q}_{|\tilde{J}_{a,b}(E)})$$

are isomorphisms. Then the reduced summand E_0 of E is the $(\tilde{J}_{a,b}(E), |\pi_2^* \mathcal{B}|, \pi_1^* \mathcal{A})$-Schwarzenberger bundle.

We can also deduce the following:

Lemma 3.3. *Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on X. Then*

$$\tilde{J}_{a,b}(E) = \tilde{J}_{a,b}(E_0),$$

where E_0 is the reduced summand of E. In particular, $J_{a,b}(E) = J_{a,b}(E_0)$.

Proof. In Lemma 1.6 we saw that $E = E_0 \oplus (\ker\varphi)^* \otimes F_1$ and that E_0 is the $(F_0, \mathcal{O}_X)$-Steiner bundle corresponding to the linear map $\varphi' : T^*/\ker\varphi \to S^* \otimes H^0(F_0^\vee)$. The statement now follows immediately, since $T_0^* = \operatorname{Im}\varphi = \operatorname{Im}\varphi'$. $\square$

We will now restrict to the case of $(1, f_0)$-jumping subspaces of an $(F_0, \mathcal{O}_X)$-Steiner bundle E on X.

Following the notation set in Definition 3.1, we will denote the set of $(1, f_0)$-jumping subspaces by $J(E)$, the set of $(1, f_0)$-jumping pairs by $\tilde{J}(E)$, and by Σ the set of 1-dimensional subspaces $A \subset S^*$ for which there exists an f_0-dimensional subspace $B \subset H^0(F_0^\vee)$ such that (A, B) is a $(1, f_0)$-jumping pair of E.

By abuse of notation, we will denote the jumping locus both as vectorial and projectivized. Therefore, the projectivization of the Segre embedding

$$\nu : G(1, S^*) \times G(f_0, H^0(F_0^\vee)) \longrightarrow G(f_0, S^* \otimes H^0(F_0^\vee))$$

is

$$\nu : \mathbb{P}(S) \times \mathbb{G}\left(f_0 - 1, \mathbb{P}\left(H^0(F_0^\vee)^*\right)\right) \longrightarrow \mathbb{G}\left(f_0 - 1, \mathbb{P}\left(S \otimes H^0(F_0^\vee)^*\right)\right).$$

It follows from the definition of jumping pair that

$$\tilde{J}(E) = \operatorname{Im}\nu \cap \mathbb{G}(f_0 - 1, \mathbb{P}(T_0)), \tag{2}$$

where $T_0 = (\operatorname{im}\varphi)^*$. We can immediately obtain a lower bound for the dimension of $\tilde{J}(E)$ by computing the expected dimension of the intersection (the case when we have a complete intersection):

$$\dim \tilde{J}(E) \geq f_0\left(t_0 - f_0 + h^0(F_0^\vee)(1 - s)\right) + s - 1,$$

where t_0 denotes the dimension of T_0.

Remark 3.4. Observe that the previous inequality implies that the dimension of the jumping variety can be negative, which means that $\tilde{J}(E)$ can be empty. In this case the corresponding Steiner bundle E cannot be a Schwarzenberger bundle.

4 The tangent space of the jumping variety

Our main purpose in this section is to obtain an upper bound for the subspace $\tilde{J}(E)$ of jumping pairs of an $(F_0, \mathcal{O}_X)$-Steiner bundle E. Our result will allow us to classify in the next section all Steiner bundles such that $\tilde{J}(E)$ has maximal dimension.

If E is defined by the sequence

$$0 \to S \otimes F_0 \to T \otimes \mathcal{O}_X \to E \to 0,$$

we will denote, as in the previous sections, the dimensions of S and T by s and t, respectively. Recall, furthermore, that Q denotes the vector bundle in (1).

Consider a jumping pair Λ in $\tilde{J}(E)$. Then Λ is an f_0-dimensional vector space that can be written as $s_0 \otimes \Gamma \subset S^* \otimes H^0(F_0^\vee) \simeq \mathrm{Hom}(H^0(F_0^\vee)^*, S^*)$. In addition, recall that the tangent space of the jumping variety at $\Lambda = s_0 \otimes \Gamma$ is the set

$$\begin{aligned} T_\Lambda \tilde{J}(E) = \quad & \left\{\psi \in \mathrm{Hom}\left(\Lambda, \tfrac{T^*}{\Lambda}\right) \mid \forall\, \varphi \in \Lambda, (\psi(\varphi))(\ker \varphi) \subset < s_0 > \right. \\ & \left. \text{and } \exists\, A \supset < s_0 > \text{ with } A \subset S^*, \dim A = 2 \text{ such that } \mathrm{Im}\, \psi(\varphi) \subset A \right\}, \end{aligned}$$

as proved in [AM14], Theorem 4.4.

Since Λ is a morphism in $\mathrm{Hom}(H^0(F_0^\vee)^*, S^*)$, we can construct three bases, $\{\lambda_i\}_{i=1}^{f_0}$ for Λ, $\{u_i\}_{i=1}^{N+1}$ for $H^0(F_0^\vee)^*$, with $N+1 := h^0(F_0^\vee)$, and $\{v_i\}_{i=1}^{s}$ for S^*, with $v_1 = s_0$, and such that

$$\lambda_i : H^0(F_0^\vee)^* \to S^*$$

$$\lambda_i(u_j) = \begin{cases} v_1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In [AM14] the authors proved that the tangent space of the jumping variety at a jumping pair can be also described as

$$T_\Lambda \tilde{J}(E) = \left\{ \psi \in \mathrm{Hom}\left(\Lambda, \frac{T^*}{\Lambda}\right) \;\middle|\; \begin{array}{l} (\psi(\lambda_i))(\ker \lambda_i) \subset \langle v_1 \rangle, \\ (\psi(\lambda_i))(u_i) \equiv (\psi(\lambda_j))(u_j) \mod v_1, \; i \neq j \end{array} \right\}. \tag{3}$$

Using this description, we are able to obtain an upper bound for the dimension of $T_\Lambda \tilde{J}(E)$ and hence an upper bound for the dimension of $\tilde{J}(E)$.

Note that when $E \cong S \otimes Q$, where Q is as in (1), $\tilde{J}(E)$ is the Segre variety. So, in this case $\dim \tilde{J}(S \otimes Q) = s - 1 + (h^0(F_0^\vee) - f_0) f_0$. For reduced non-TSB Steiner bundles on X we get:

Theorem 4.1. *Let E be a reduced non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X and let $\sigma : X \to \mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^\vee)))$ be the natural morphism. For every $\Lambda \in \tilde{J}(E)$,*

$$\dim T_\Lambda \tilde{J}(E) \leq f_0 \left(t - \dim \sigma(X) - f_0 s + 1\right).$$

In particular, $\dim \tilde{J}(E) \leq f_0 \left(t - \dim \sigma(X) - f_0 s + 1\right)$.

Proof. Recall that the tangent space of the jumping variety at a jumping point is a vector subspace of $\mathrm{Hom}\left(\Lambda, \frac{T^*}{\Lambda}\right)$. We will prove the statement by defining independent elements in $\mathrm{Hom}\left(\Lambda, \frac{T^*}{\Lambda}\right)$ which are also independent modulo $T_\Lambda \tilde{J}(E)$. In order to do so, we will look for morphisms in $\frac{T^*}{\Lambda}$ which do not satisfy the conditions in (3).

Let us consider the following diagram

$$\begin{tikzcd} T^* \arrow[r, hook, "\varphi"] \arrow[d] & \mathrm{Hom}\left(H^0(F_0^\vee)^*, S^*\right) \arrow[d] \\ \frac{T^*}{\Lambda} \arrow[r, "\varphi_1"] & \mathrm{Hom}\left(H^0(F_0^\vee)^*, \frac{S^*}{\langle v_1 \rangle}\right) \end{tikzcd}$$

Observe that the linear map φ_1 also defines an $(F_0, \mathcal{O}_X)$-Steiner bundle $\tilde{E}$.

We want to estimate the dimension of the image of φ_1. We first note that $\operatorname{Hom}\left(H^0(F_0^\vee)^*, \frac{S^*}{\langle v_1\rangle}\right)$ can be identified with the vector space of global sections of the bundle

$$\frac{S}{\langle v_1\rangle^*} \otimes \mathcal{U}^\vee \longrightarrow \mathbb{G}(f_0-1, \mathbb{P}(H^0(F_0^\vee))).$$

Then, the image of φ_1 can be identified with the space of global sections of the restriction of the previous bundle to $\sigma(X)$, where σ is the natural morphism $\sigma : X \longrightarrow \mathbb{G}(f_0-1, \mathbb{P}(H^0(F_0^\vee)))$. Consider the bundle morphism

$$\begin{tikzcd}
\mathcal{O}^\alpha_{\sigma(X)} \arrow[rr, "g"] \arrow[dr] & & \frac{S^*}{\langle v_1\rangle} \otimes \mathcal{U}^\vee_{|\sigma(X)} \arrow[dl] \\
& \sigma(X) &
\end{tikzcd}$$

By Porteous' formula, the morphism g is not surjective if $\alpha \le \dim\sigma(X) + (s-1)f_0 - 1$. We thus have at least $\dim\sigma(X) + (s-1)f_0$ independent morphisms in $\operatorname{Im}\varphi_1$ (and hence $\mu_1, \ldots, \mu_{\dim\sigma(X)+(s-1)f_0}$ morphisms in T^*/Λ).

Let us construct morphisms $\psi_{i,j}$ belonging to $\operatorname{Hom}\left(\Lambda, \frac{T^*}{\Lambda}\right)$ in the following way:

$$\psi_{i,j} : \Lambda \to T^*/\Lambda$$

$$\psi_{i,j}(\lambda_k) = \begin{cases} \mu_j & \text{if } i = k, \\ 0 & \text{if } i \ne k, \end{cases}$$

for $i = 2, \ldots, f_0$ and $j = 1, \ldots, \dim\sigma(X)+(s-1)f_0$. Then we have $(f_0-1)(\dim\sigma(X)+(s-1)f_0)$ of such independent morphisms and it can be easily verified that they do not satisfy any of the conditions in the definition of the tangent space.

Let us now extend the previous diagram to the following one:

$$\begin{tikzcd}
T^* \arrow[r, hook, "\varphi"] \arrow[d, two heads] & \operatorname{Hom}\left(H^0(F_0^\vee)^*, S^*\right) \arrow[d, two heads] \\
\frac{T^*}{\Lambda} \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"] & \operatorname{Hom}\left(H^0(F_0^\vee)^*, \frac{S^*}{\langle v_1\rangle}\right) \arrow[d, two heads] \\
& \operatorname{Hom}\left(\ker\lambda_1, \frac{S^*}{\langle v_1\rangle}\right)
\end{tikzcd}$$

We want to estimate the dimension of the image of φ_2. As before, $\operatorname{Hom}\left(H^0(F_0^\vee)^*, \frac{S^*}{\langle v_1\rangle}\right)$ can be identified with the vector space of global sections of the bundle $\frac{S}{\langle v_1\rangle^*} \otimes \mathcal{U}^\vee$ on $\mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^\vee)))$, and the image of φ_1 can be identified with the global sections' vector space of the restriction of this bundle to $\sigma(X)$. Finally, if we take the subgrassmannian $\mathbb{G}(f_0-1, \mathbb{P}((\ker\lambda_1)^*))$, the image of φ_2 can be identified with the space of global sections of the vector bundle

$$\frac{S}{\langle v_1\rangle} \otimes \mathcal{U}^\vee_{|(\sigma(X)\cap\mathbb{G}(f_0-1,\mathbb{P}(\ker\lambda_1)^*))} \longrightarrow \sigma(X) \cap \mathbb{G}(f_0-1, \mathbb{P}(\ker\lambda_1)^*).$$

Denote $W = \sigma(X) \cap \mathbb{G}(f_0 - 1, \mathbb{P}(\ker \lambda_1)^*)$ and consider the bundle morphism

$$\begin{tikzcd} \mathcal{O}_W^{\alpha} \arrow[rr, "g"] \arrow[dr] & & \frac{S^*}{\langle v_1 \rangle} \otimes \mathcal{U}^{\vee}_{|W} \arrow[dl] \\ & W & \end{tikzcd}$$

Using Porteous' formula again we see that the morphism g is not surjective if $\alpha \leq \dim \sigma(X) + (s-2)f_0 - 1$. This means that we can find at least $\dim \sigma(X) + (s-2)f_0$ independent morphisms in the image of φ_2 and hence $\tilde{\mu}_j \in T^*/\Lambda$, with $j = 1, \ldots, \dim \sigma(X) + (s-2)f_0$. So we can define a new set $\{\psi_{1,j}\}_{j=1}^{\dim \sigma(X)+(s-2)f_0} \subset \mathrm{Hom}\left(\Lambda, \frac{T^*}{\Lambda}\right)$ by

$$\psi_{1,j} : \Lambda \to T^*/\Lambda$$

$$\psi_{1,j}(\lambda_k) = \begin{cases} \tilde{\mu}_j & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}$$

and not satisfying the conditions defining the tangent space.

We have thus just constructed a minimum of $f_0(\dim \sigma(X) - f_0 + f_0 s - 1)$ linearly independent morphisms $\psi_{i,j}$ in the complementary vector space of $T_\Lambda \tilde{J}(E)$. In particular, $\dim T_\Lambda \tilde{J}(E) \leq f_0(t - f_0) - f_0(\dim \sigma(X) - f_0 + f_0 s - 1) = f_0 \left(t - \dim \sigma(X) - f_0 s + 1\right)$ and the theorem is proved. $\square$

5 The classification

In this section we classify all $(F_0, \mathcal{O}_X)$-Steiner bundles whose jumping variety has maximal dimension. In particular, we prove that they are always Schwarzenberger bundles.

Let E be an $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X, where F_0 is a rank f_0 vector bundle on X. Recall again that we have (see (1)) an exact sequence given by

$$0 \longrightarrow F_0 \longrightarrow H^0(F_0^{\vee})^* \otimes \mathcal{O}_X \longrightarrow Q \longrightarrow 0.$$

Moreover, following the notation in Lemma 3.2, let π_1 and π_2 denote respectively, the projections from $\tilde{J}(E)$ to $G(1, S^*)$ and $G(f_0, H^0(F_0^{\vee}))$. Then $\Sigma(E) = \pi_1(\tilde{J}(E))$ and $J(E) = \pi_2(\tilde{J}(E))$.

We will first state the theorem. The rest of the section will be devoted to its proof.

Theorem 5.1. *Let E be a reduced $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X such that the jumping locus $\tilde{J}(E)$ has maximal dimension. Suppose that the morphism $\sigma : X \longrightarrow \mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^{\vee})))$ is generically finite.*

If E is TSB then $E \simeq S \otimes Q$.

If E is non-TSB then it is one of the following.

i) A Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^N}(1))|, \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(s-1))\right).$$

In this case $f_0 = 1$, $\tilde{J}(E)$ is a rational normal curve and the natural projections are

$$\tilde{J}(E) \xrightarrow{\pi_1} \Sigma(E) \simeq \mathbb{P}^1$$

$$\tilde{J}(E) \xrightarrow{\pi_2} \mathbb{P}^N$$

with $N = h^0(F_0^{\vee}) - 1$.

ii) A Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{U}^\vee)|, \pi_1^*(\mathcal{O}_{\mathbb{P}(S)}(1))\right).$$

In this case $s \leq f_0+1$, $\tilde{J}(E)$ is the projectivization of a Grassmannian bundle constructed from a rational normal scroll and the natural projections are

$$\tilde{J}(E) \xrightarrow{\pi_1} \Sigma(E) \simeq \mathbb{P}(S)$$
$$\tilde{J}(E) \xrightarrow{\pi_2} \mathbb{G}\left(f_0-1, \mathbb{P}(H^0(F_0^\vee))\right).$$

iii) A Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{U}^\vee)|, \mathcal{O}_{\tilde{J}(E)}(1))\right).$$

In this case $f_0 > 1$, $\tilde{J}(E) \simeq \Sigma(E)$ and we have the natural projection

$$\tilde{J}(E) \xrightarrow{\pi_2} \mathbb{G}\left(f_0-1, \mathbb{P}(H^0(F_0^\vee))\right).$$

iv) A Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|, \pi_1^*(\mathcal{O}_{\Sigma(E)}(1))\right).$$

In this case $f_0 = 1$, $s \geq 3$, $\tilde{J}(E)$ is a rational normal scroll and the natural projections are

$$\tilde{J}(E) \xrightarrow{\pi_1} \Sigma(E)$$
$$\tilde{J}(E) \xrightarrow{\pi_2} J(E) \simeq \mathbb{P}^1.$$

v) A Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1))|, \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))\right).$$

In this case $f_0 = 1$, $s = 3$, $\tilde{J}(E)$ is a Veronese surface and the natural projections are

$$\tilde{J}(E) \xrightarrow{\pi_1} \Sigma(E) \simeq \mathbb{P}^2$$
$$\tilde{J}(E) \xrightarrow{\pi_2} J(E) \simeq \mathbb{P}^2.$$

Remark 5.2. If the morphism $\sigma : X \longrightarrow \mathbb{G}(f_0-1, \mathbb{P}(H^0(F_0^\vee)))$ is surjective, then an $(F_0, \mathcal{O}_X)$-Steiner bundle E on X induces a $(\mathcal{U}, \mathcal{O}_{\mathbb{G}})$-Steiner bundle $\bar{E}$ on the Grassmannian, with $E = \sigma^*(\bar{E})$. Moreover, $\tilde{J}(E)$ has maximal dimension if and only if $\tilde{J}(\bar{E})$ has maximal dimension, according to the respective bounds. Therefore, when σ is surjective and $\tilde{J}(E)$ has maximal dimension, all Steiner bundles on X are Schwarzenberger bundles given by the pullback of the corresponding Schwarzenberger bundle on $\mathbb{G}(f_0-1, \mathbb{P}(H^0(F_0^\vee)))$, classified in [AM14]. Therefore, we can suppose, from now on, that σ is not surjective.

Let E be a reduced $(F_0, \mathcal{O}_X)$-Steiner bundle.

The first statement in the theorem, when $E \cong S \otimes Q$, follows from Proposition 1.9, the fact that $\tilde{J}(E) = \mathbb{P}(S) \times \mathbb{G}\left(f_0-1, \mathbb{P}\left(H^0(F_0^\vee)^*\right)\right)$ and Lemma 3.2, *(iii)*.

Suppose now that E is non-TSB. Take $\Lambda = s_0 \otimes \Gamma \in \tilde{J}(E)$. We have the following commutative diagram:

$$\begin{tikzcd} T^* \arrow[r, hook, "\varphi"] \arrow[d] & S^* \otimes H^0(F_0^\vee) \arrow[d, "pr\otimes id"] \\ \frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{\langle s_0 \rangle} \otimes H^0(F_0^\vee) \end{tikzcd} \tag{4}$$

The morphism φ' defines a new $(F_0, \mathcal{O}_X)$-Steiner bundle E', whose defining vector spaces S' and T' have dimension $s-1$ and $t-f_0$, respectively. Let E'_0 be its reduced summand. Iterating this process, which we will call *induction technique*, we will see that eventually we always arrive to a known case of a reduced $(F_0, \mathcal{O}_X)$-Steiner bundle that can be described as a Schwarzenberger bundle.

Moreover, we also have a diagram

$$\begin{tikzcd} & \tilde{J}(E) \arrow[dl, "\pi_1"'] \arrow[d] \\ \Sigma(E) \arrow[d, "pr_{s_0}"'] & \tilde{J}(E'_0) \arrow[dl, "\pi'_1"'] \\ \Sigma(E'_0) & \end{tikzcd} \tag{5}$$

Properties in Theorems 4.3 and 4.4 in [AM14] will still hold. In particular, $\Sigma(E)$ will always be a minimal degree variety, $\tilde{J}(E'_0)$ is birational to $\Sigma(E)$ and the morphism pr_{s_0} is a projection from an inner point $s_0 \in \Sigma(E)$. Hence $\dim \Sigma(E'_0) \leq \dim \Sigma(E) \leq \dim \Sigma(E'_0) + 1$.

The generic fibers of π_1 and π'_1, and the further first component projections given by the induction technique, have respectively dimension either 0 or at least f_0.

We will next prove statements $i) - v)$ by analyzing two cases. First, we will look at the special case when $s \geq f_0 + 1$ and prove $ii)$. Second, we will look at the general case, by studying the cases when $f_0 = 1$ and $f_0 > 1$, and prove $i)$, $iii) - v)$.

5.1 The case $s \leq f_0 + 1$

Suppose that $s \leq f_0 + 1$ and consider a non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle E on X whose jumping variety $\tilde{J}(E)$ has maximal dimension, i.e.

$$\dim \tilde{J}(E) = f_0 \left(t - \dim X - s f_0 + 1\right).$$

Proposition 5.3. *Let E be a non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle E on a smooth projctive variety X whose jumping variety $\tilde{J}(E)$ has maximal dimension. If $s \leq f_0 + 1$ then the projection $\pi_1 : \tilde{J}(E) \longrightarrow \mathbb{P}(S)$ is surjective.*

Proof. Suppose that π_1 is not surjective, which implies that $\dim \Sigma(E) < s - 1 \leq f_0$. Hence, for each $s_0 \in \Sigma(E)$, we have that $\dim \pi_1^{-1}(s_0) = \dim \tilde{J}(E) - \dim \Sigma(E) > f_0(t - \dim X - s f_0)$. Since $\pi_1^{-1}(s_0) \simeq G(f_0, \left(\langle s_0 \rangle \otimes H^0(F_0^\vee)\right) \cap T^*)$, we get that $\dim \left(\left(\langle s_0 \rangle \otimes H^0(F_0^\vee)\right) \cap T^*\right) > t - \dim X +$

$(s-1)f_0$. Consider the following diagram, obtained by taking a jumping pair $\Lambda = s_0 \otimes \Gamma \in \tilde{J}(E)$:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \langle s_0\rangle \otimes H^0(F_0^\vee) \arrow[d] \\
T^* \arrow[ur, "\varphi''"] \arrow[r, hook, "\varphi"] \arrow[d, "\pi"'] & S^* \otimes H^0(F_0^\vee) \arrow[d] \\
\frac{T^*}{\Lambda} \arrow[r, "\varphi'"] & \frac{S^*}{\langle s_0\rangle} \otimes H^0(F_0^\vee) \arrow[d] \\
 & 0
\end{tikzcd}$$

Being $\mathrm{Im}\ \varphi'' \simeq \big(\langle s_0\rangle \otimes H^0(F_0^\vee)\big) \cap T^*$, we have $\dim \mathrm{Im}\, \varphi' = \dim \mathrm{Im}\, \varphi - \dim \mathrm{Im}\, \varphi'' < \dim X + (s-1)f_0$.

Recall that $\dim X < (s-1)\cdot \mathrm{rk}(Q)$. Since φ' also defines a Steiner bundle E', it follows from Theorem 1.11 that $\mathrm{rk}(E') \geq \dim X$ and therefore $\dim \mathrm{Im}\, \varphi' \geq \dim X + (s-1)f_0$, leading us to contradiction.

$\square$

Remark 5.4. Since E is a non-TSB Steiner bundle, one deduces from the previous proposition that under the given hypothesis all fibers $\pi_1^{-1}(s_0)$, for each $s_0 \in \mathbb{P}(S)$, have the same dimension, namely $f_0(t - \dim X - sf_0 + 1) - (s-1)$. Hence, when $s \leq f_0 + 1$, the jumping variety $\tilde{J}(F)$ is the projectivization of a Grassmannian bundle constructed from a rational scroll on $\mathbb{P}(S)$. In particular, it is smooth.

Consider the two natural projections $\tilde{J}(E) \xrightarrow{\pi_1} \mathbb{P}(S)$ and $\tilde{J}(E) \xrightarrow{\pi_2} \mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^\vee)^*))$. By Lemma 3.2 we obtain the following result.

Theorem 5.5. *Let E be a reduced non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X such that $\tilde{J}(E)$ has maximal dimension and $\sigma : X \longrightarrow \mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^\vee)))$ is generically finite. If $s \leq f_0 + 1$ then E is a Schwarzenberger bundle defined by the triple*

$$(\tilde{J}(E), |\pi_2^*(\mathcal{U}^\vee)|, \pi_1^*(\mathcal{O}_{\mathbb{P}(S)}(1))),$$

where $\mathcal{U} \longrightarrow \mathbb{G}(f_0 - 1, \mathbb{P}(H^0(F_0^\vee)^*))$ denotes the rank f_0 universal bundle.

This proves part (ii) of Theorem 5.1.

5.2 The general case

Recall diagram (5) and the fact that each first component projection has dimension either 0 or at least f_0. We will thus divide the classification in the cases $f_0 = 1$ and $f_0 > 1$.

5.2.1 Case $f_0 = 1$

If $f_0 = 1$ and $s = 2$ we have already proved in Theorem 5.5 that $\tilde{J}(E)$ is a rational normal scroll, $\Sigma(E) = \mathbb{P}^1$ and, moreover, E is Schwarzenberger bundle.

It was proved in [AM14] that if π_1 is not birational then each further projection on the first component, given by the iteration process, is not birational.

Let us consider first the case when all projections π_1 are birational. Applying the induction technique until the case $s = 2$, we obtain that $\tilde{J}(E)$ is a rational normal curve because it is birational to $\mathbb{P}^1$. So, considering the following diagram

$$\begin{tikzcd}
\tilde{J}(E) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{P}(H^0(F_0^\vee)) =: \mathbb{P}^N \\
\mathbb{P}^1 &
\end{tikzcd}$$

we get, by Lemma 3.2, that E is Schwarzenberger bundle on X defined by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^N}(1))|, \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(s-1))\right).$$

Let us suppose now that the birationality is broken at some step of the induction. Without loss of generality, we can focus on such step and we are in the following situation ($+1$ will denote that the fiber is one dimensional):

$$\begin{tikzcd}
 & & \tilde{J}(E) \arrow[dl, "bir"'] \arrow[d, "bir"] \\
\text{step } s & \Sigma(E) \arrow[r, dashed, dash, "bir"] \arrow[d, "+1"'] & \tilde{J}(E_0') \arrow[dl, "+1"'] \arrow[dd, dashed] \\
\text{step } s-1 & \mathbb{P}^{s-2} \simeq \Sigma(E_0') \arrow[dd, dashed] & \\
 & & \tilde{J}(\bar{E}_0) \arrow[dl, "+1"'] \\
\text{step } 2 & \mathbb{P}^1 &
\end{tikzcd}$$

From the classification of the case $s = 2$, we have that $\tilde{J}(\bar{E}_0)$ is a surface. One can prove that it is a quadric surface, which implies that $J(\bar{E}) \simeq \mathbb{P}^1$. Therefore, also $J(E) \simeq \mathbb{P}^1$ and $\tilde{J}(E)$ is a rational normal scroll over $\mathbb{P}^1$. Denoting as usual $\pi_1 : \tilde{J}(E) \longrightarrow \Sigma(E)$ and $\pi_2 : \tilde{J}(E) \longrightarrow \mathbb{P}^1$, we obtain by Lemma 3.2 that, if $s \geq 4$, E is the Schwarzenberger bundle defined by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|, \pi_1^*(\mathcal{O}_{\Sigma(E)}(1))\right).$$

Let us consider the only left case when $f_0 = 1$, that is the case $s = 3$, described by the following diagram:

$$\begin{tikzcd}
 & & \tilde{J}(E) \arrow[dl, "bir"'] \arrow[d, "bir"] \arrow[dr, "\pi_2"] & \\
\text{step } 3 & \mathbb{P}^2 \arrow[d, "pr_{s_0}"'] & \tilde{J}(E_0') \arrow[dl, "+1"'] \arrow[dr, "+1"] & J(E) \arrow[d, equal] \\
\text{step } 2 & \mathbb{P}^1 & & J(E_0')
\end{tikzcd}$$

We have already proved that $\tilde{J}(E_0')$ is a rational normal scroll of dimension 2, and, as shown in [AM14], the variety $\tilde{J}(E)$ can be either a Hirzebruch surface or a Veronese surface.
In the first case E is the Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|, \pi_1^*(\mathcal{O}_{\Sigma(E)}(1))\right).$$

In the second case we have that E is the Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{O}_{\mathbb{P}^2})|, \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))\right).$$

This proves parts (i), (iv) and (v) of Theorem 5.1.

5.2.2 Case $f_0 > 1$

Let us study now the case with $f_0 > 1$. Looking at the Diagram (5), we recall it is impossible to get $\dim \Sigma(E) = \dim \Sigma(E_0') + 1$, because we have already noticed that the fiber of the projections of type π_1 has dimension either zero or greater equal than f_0, which would lead to contradiction. This means that all the projections involved in the diagram are birational. Recall the following lemma proved in [AM14], that also applies in the current situation.

Lemma 5.6. *Let E be a reduced non-TSB $(F_0, \mathcal{O}_X)$-Steiner bundle on a smooth projective variety X and $\tilde{J}(E)$ its jumping locus. Suppose that $\tilde{J}(E)$ is birational to $\Sigma(E)$ and, fixed a jumping pair $s_0 \otimes \Gamma$, consider the first step of the induction (Diagram (5)). If the morphism π_1' is an isomorphism then also π_1 is an isomorphism.*

The combination of this lemma and the birationality of the projections imply that $\tilde{J}(E) \simeq \Sigma(E)$, and so E is the Schwarzenberger bundle given by the triple

$$\left(\tilde{J}(E), |\pi_2^*(\mathcal{U}^\vee)|, \mathcal{O}_{\tilde{J}(E)}(1)\right),$$

proving (iii) in the theorem.

We have completed the study of all possible cases and we have thus proved Theorem 5.1.

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