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Anticipating Price Exuberance

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Abstract

This thesis develops a new econometric mechanism to predict speculative bubbles. Along history price exuberance has been an important source of economic recessions. Thus it is very important for regulators and policy makers to possess an ex ante tool capable of anticipating such events, enabling them to act accordingly. The main objective is infer about a significant probability of exuberance at least one step ahead of a bubble peak. When compared with other approaches this provides a combination of asset pricing equilibrium and non stationarity analysis. Using the former components as inputs of a dynamic probit specification, one develops a mechanism where the fundamentals contained in the asset abnormal return judge the explosiveness in the price. Simulations reveal good statistical properties and the mechanism is able to successfully anticipate the "technological bubble" observed in the 90's estimating probabilities higher than 85% five periods before the bubble peak. (Speculative Bubbles, Asset Pricing, Macroeconometrics, Adaptive Learning) (JEL C22, G17)

Resumo

Nesta dissertação é proposto um novo mecanismo econométrico para a previsão de bolhas especulativas. Ao longo da história a especulação financeira tem se revelado uma causa importante de recessões económicas. Neste sentido, é muito importante para as instituições reguladoras a obtenção de uma ferramenta ex ante capaz de antecipar tais eventos. O principal objetivo deste mecanismo é inferir sobre a probabilidade de existência de uma bolha especulativa, pelo menos, um período antes do pico dessa mesma bolha. Quando comparado com outras abordagens o procedimento proposto proporciona uma inovação que se baseia na combinação entre modelos financeiros de equilíbrio e de análise macroeconométrica. Usando os resultados de cada um dos modelos anteriores como variáveis independentes num modelo Probit dinâmico, obtém-se um mecanismo em que os fundamentais contidos no retorno anormal conseguem diferenciar se o tipo de aceleração que se observa no preço se deve de facto a uma bolha especulativa. As simulações revelam boas propriedades estatísticas e o mecanismo é capaz de antecipar com sucesso a "bolha tecnológica" observada na década de 90, estimando uma probabilidade superior a 85 % cinco períodos antes do pico da bolha. (Bolhas Especulativas, Modelos Financeiros de Equilíbrio, Macroeconometria, Aprendizagem Adaptativa) (JEL C22, G17)

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1 Introduction & Motivation

Since the first documented event of price exuberance, the "Tulipomania"[1], that economists have been trying to understand how investors react in such market conditions. Modern Finance theory is grounded on the concept of market efficiency Fama (1970), in which market price reflects all the available information. In fact standard economic theory advocates the price system has the most efficient mean for consumers utility maximization. Considering the existence of speculative bubbles it's easy to understand that they constitute an exception to the former reasoning, and by consequence they need to be properly identified. The price of stock today is the infinite sum of all their expected future dividends, which means the stock will only worth a certain amount of money if the underlying fundamentals are able to generate a certain value of dividends. When price exuberance ends one perceives a considerable gap between the observed price and the underlying fundamentals supporting that same price, giving rise to the creation of speculative bubbles. In line with this it's very important for the market as a whole, to identify this type of event when it is happening and not in an ex post analysis, because in this situation **the price alone cannot provide the best decision input** for the consumer perform his utility maximization.

As presented in the next section the challenge inherent to this exercise, is the proper identification of fundamentals in the observed price acceleration, so that the bubble can be properly anticipated and both agents and policy makers can act accordingly. Developing the presented mechanism also pushes the literature one step further towards the elimination of a persistent gap which is the negligence of unobserved fundamentals.

This dissertation is structured as follows: Section (2) I discuss the most influential literature on speculative bubbles. Section (3) presents a review

[1] This speculative event happened in Holland in 1637 and was characterize by the preposterous increase of the tulip bulbs price, in which the most collectible bulbs reached prices equivalent of luxury real states.

of the most important concepts underlying the new anticipation procedure ranging from asset pricing theory to macroeconometrics. Section (4) provides a complete description of the new procedure and its contribution towards the established literature. Section (5) offers a single, highly detailed, simulation replica of the new procedure as well the Monte Carlo results obtained from one hundred replicas. Finally in Section (6) the new procedure is applied to Apple and Nasdaq Composite monthly data from 1990 to 2014.

2 Literature Review

The literature concerning asset bubbles is quite extensive and the existent approaches can be roughly separated between theoretical macroeconomics, econometric modeling or a tune of both. The idea of extrinsic variables leading asset prices is somehow controversial, most authors start with a common framework and usually arrive at significant different conclusions about bubble's existence.

The emergence of rational expectations in the seventies motivated literature like (Shiller, 1981), where the author derives theoretical limits for asset prices variation considering the underlying dividends (fundamentals). An empirical application with S&P 500 data, revealed an exaggerated volatility, which the author does not link to price exuberance, implying the poor capability of the general asset pricing equation as theorized in Gurkaynak (2005). Using the same framework, Blanchard (1979), pioneered the "periodically collapsing bubbles" stochastic processes, by allowing the common exponential growth process to periodically collapse according to some fixed probability. Their importance would only come to be recognized latter down the way, mainly due to robust theoretical restrictions against price exuberance. The most notorious examples are Tirole (1982) in which bubbles are ruled out through the violation of the transversality condition and Diba & Grossman (1988b) where investors' free asset disposal leads to demonstrate the following reasoning: If a bubble exists then it should had started in the first day of trading and by consequence when it bursts it can not re-initiate.. Another important aspect of an asset price is given in Hamilton (1986)where he demonstrates the determining of price bubbles being caused by explosive behavior in underlying unobserved fundamentals, like taxes, which are anticipated by economic agents but not by statisticians.

Pioneering the simulation analysis of price exuberance, Diba & Grossman (1988a) built two arguments against the existence of rational bubbles: First if dividends and prices are stationary in first differences then bubbles are not present, otherwise even if differentiated n times the price process will still have explosive behavior. Secondly if prices and dividends are co-integrated then bubbles are not present. The first test corresponds to an application of Dickey & Fuller (1979), as the second was developed in Engle & Granger (1998). It is emphasized in the paper that even if these tests reveal the presence of bubbles there is always the case of explosive unobserved fundamentals as pointed out in Hamilton (1986). In response to this line of argumentation, a very strong result is presented in Evans (1991), the author uses simulation to demonstrate the reduced power of unit root and co-integration tests, when periodically collapsing bubbles alla Blanchard (1982) are present in the data generating process (DGP), instead of standard bubble innovation process used by Diba & Grossman (1988a). The intuition for this result is quite simple: Since there are several bubble collapses in the simulated sample, the process will "appear" to be stationary and the discriminatory power of the former tests do not reveal the explosiveness presented in the data, even if the former explosive process is present. This paper sets the literature in an endeavor for finding proper econometric tests towards the detection of asset price exuberance. In an attempt to solve the power issues presented in Evans (1991), a modified version of the basic ADF regression is developed in Hall et al. (1999), by incorporating a Markov-Switching mechanism where the parameters are allowed to change from a stationary to an explosive process imitating the behavior of periodically collapsing bubbles. Using the same stochastic process as in Evans (1991), a simulation test reveals that in 65% of the cases the test can correctly identify the correct regimes, where as the standard ADF failed to identify any explosiveness. In line with this there is also Phillips (2012), in which he developed not only a recursive Augmented Dickey Fuller statistic (SADF) but also a date stamp estimator for the eruption and collapsing period of the bubble. The test is performed for different sub-samples and in each one, the mildly explosive null hypothesis ($\rho \leq 1$) is tested against the explosive alternative ($\rho > 1$). The application to simulated data as in Evans (1991) revealed good power properties dealing with periodically collapsing bubbles comparing with the standard ADF, nevertheless it is still dependent on the probability of collapse. Applied to the NASDAQ Composite index the test not only revealed explosive behavior, but it could date stamp reasonably well the "dotcom" exuberance. Alongside the former approach other tests have been develop to analyze the presence of bubbles in the data. In Homm & Jorg (2012) they are all reviewed and compared. Using a time varying AR(1) process the authors test for random walk under the null ($\rho_t = 1 \forall t$) against random walk to explosiveness switching under the alternative ($\rho = 1$ until $[\tau^*\mathrm{T}]$ then $\rho > 1$ from $[\tau^*\mathrm{T}]$ to T, where T is the sample size and $\tau^* \in$ [0,1]). Five different statistics are analyzed: a modified version of the one presented in Bhargava (1986), the Kim statistic developed in Kim (2000), the supBT which is also a modified version of the one presented in Busetti & Taylor (2004), the formerly presented supDF developed in Phillips (2012) and the DFC statistic which is a modified version of the one presented in Dickey & Fuller (1979), allowing for structural breaks in data. To do the comparisons a simulated sample is generated using the same DGP as in Evans (1991) revealing the most powerfull statistic as the supDF, followed by supDFC and supBT. According to the author, this result is intuitive because in its construction the supDF does not include bubble collapses in the testing sub-sample as the others do. Eliminating collapses from the sample the supDFC and supBT would outperform the supDF. Considering the data stamping estimation, this paper finds the supDFC estimator the most reliable one, as the supDF tends to estimate the breaking dates to early and is less efficient. An improved version of the supDF and a new date stamping methodology is presented in Phillips et al. (2013). The main novelty is the rolling sub-sample, meaning that instead of a fixed point in the beginning of the sample and the other one moving forward, now both points move, which constitutes a generalization of the test presented in Phillips (2012), originating the GSADF(Generalized Supreme Augmented Dickey Fuller). Besides the improved size distortions, the GSADF has a greater discriminatory power than its predecessor. Performing the same simulations as in Evans (1991) the authors found that the new GSADF is quite less sensitive to bubble collapses in the data, as the SADF only found bubbles in a collapse free truncated version of the sample. In an empirical application with the S&P 500 price-dividend ratio from 1871 to 2010 both tests revealed explosive sub-periods but only the GSADF statistic could date stamp all episodes of asset price exuberance since 1900.

The main drawback in the literature's approaches is the lack of a complete framework capable of explaining the bubble phenomena. In fact all the former approaches represent different ways of capturing price explosion and not necessarily price exuberance. The model presented in Branch & Evans (2011) provides a complete theoretical structure to explain the bubble phenomena. The beginning of exuberance occurs when investors under estimate price variance, which eventually overcomes a certain threshold and forces investors to sell off the asset due to utility loss. The former dynamic, occurs as the investor constantly re-estimates his learning rule Evans & Honkapohja (2001) using a recursive least squares algorithm Young (2011).

Another alternative to bubble testing is given in Fry (2014), in which the price is modeled by a differential equation constantly comparing the current price with the long term one. The authors also pioneered a generalization approach capable of estimating bubble contagion between different markets. An empirical application to bitcoin data from 2010 until 2014 revealed that the currency intrinsic value is zero. Cheah & Fry (2015)

3 Theoretical Foundations

In the next sections, I review some established literature about asset pricing, rational bubbles and macroeconometrics. Section (3.1) serves as the base platform for the asset pricing theory from which one obtains the general pricing equation. Applying the Rational Expectations theory, discussed in Section (3.2), one arrives at the main pricing equation in bubbles literature. In Section (3.3) the possibility of speculative bubbles under rational expectations is fully explored along with different bubble processes. Section (3.4) exposes the Learning theory as the generalization of Rational Expectations and it's higher suitability regarding the bubble phenomena. In Section (3.5) and (3.6) is present the core explanation of the asset pricing model and the non stationary statistic utilized in the new mechanism.

3.1 Consumption Based Asset Pricing

The pricing equation presented in Cochrane (2005)(p.6) is a general specification of all asset pricing theory, capable of yielding a price for any asset class in the economy. It is the result of the following inter temporal maximization:

$$\begin{gathered}\max_{\xi_t}\{u(C_t)+\beta E_t[u(C_{t+1})]\} \\ s.t \quad \begin{cases} C_t = e_t - P_t\xi_t \\ C_{t+1} = e_{t+1} - x_{t+1}\xi_t \end{cases}\end{gathered} \tag{1}$$

Where P_t is the asset price at time, β represents the investor's subjective discount, e_t is the investor's endowment and E_t represents the expectation conditioned on all available information at time t, which can also be represented by $E_t[.] = E[.|\Phi_t]$, where Φ_t represents the available information at time t.

At each point in time the investor maximizes his expected utility by choosing how much of the asset (ξ_t) he will hold from one period to the next ($t \rightarrow t+1$). The level of consumption in both periods (c_t and c_{t+1}) depends on the investment level (ξ_t) and the asset payoff (x_{t+1}). The former problem will yield as solution the following pricing equation, whose derivation is fully outline in Section (9.2.1):

$$P_t = E_t[m_{t+1}x_{t+1}] \tag{2}$$

Where $m_{t+1} \equiv \beta\frac{u'(c_{t+1})}{u'(c_t)} = \beta\frac{\frac{du(c_t)}{dc_t}}{\frac{du(c_{t+1})}{c_{t+1}}}$, represents the *Stochastic Discount Factor* (SDF) and x_{t+1} represents a generic asset payoff. Intuitively the equation is telling the investor what is the price he should pay today (period t), according to his expected marginal utility growth $E_t[\frac{u'(c_{t+1})}{u'(c_t)}]$ and the asset expected payoff, $E_t[x_{t+1}]$. All properly discounted at the subjective rate (β), which reflects the investor's impatience to postpone his consumption to period ($t+1$).

It is quite clear that no assumptions were made concerning the investor's utility function ($u(.)$) as well as the asset class, giving to equation (2) the proficiency to **link all asset pricing theory**, from Stocks to Options, in one simple equation completely founded in Microeconomic rational behavior.

3.2 Rational Expectations & Asset Pricing

Having obtained the general asset pricing equation (2), we can access the most widely used equation in bubbles testing literature, by simply assuming a linear structure of the utility function, the derivation is presented in Section (9.2.2):

Assuming the linearity of the investor's utility function, implying: $u'(c_t) = u'(c_{t+1}) = c \in \Re$*, we obtain the asset pricing equation presented in Lucas (1978).*

$$P_t = (\frac{1}{1+r})E_t[P_{t+1} + D_{t+1}] \tag{3}$$

The price specification given by (3), holds the same strong intuition as the general pricing equation (2) but now is applied to a general stock, so the future payoff (x_{t+1}) will be the future price (p_{t+1}) plus future dividends (d_{t+1}).

One can see, from equation (3), that the investor should have a law to produce his expectations.

We assume **rational expectations**, as developed in (Muth, 1967), which, in an intuitive manner, states that our investor's expectation about tomorrow's payoff, will be **exactly the same** as the relevant economic model used to explain that reality. In this particular case, this model is given by equation (3). Furthermore the investor calculates a conditional expectation, in which he makes use of **all publicly available information**, mathematically this can be represented by $E_t[P_{t+1}] \equiv E[P_{t+1}|\Phi_t]$, in which Φ_t corresponds to a set containing all the available information. To solve (3), using rational expectations, one can use a number of different methods (see Section (9.2.5)). The most simple one is **iterating forward n times** the difference equation (3), and the result will be:

$$P_t = \sum_{i=0}^{n}[(\frac{1}{1+r})^i \times E_t[D_{t+i}]] + (\frac{1}{1+r})^n E_t\{P_{t+n}\} \tag{4}$$

However, (4), is still dependent on the discounted expected price n periods ahead from today, $(\frac{1}{1+r})^n E_t[P_{t+n}]$, which may be the cause for **speculative bubbles**[2] as demonstrated in the next section. For that reason the model is completely closed only when the equation (3) is iterate forward by an infinite number of times ($n \rightarrow +\infty$), this will yield the *fundamental price of the stock*:

$$F_t = \sum_{i=1}^{+\infty}[(\frac{1}{1+r})^i \times E_t[D_{t+i}]] \tag{5}$$

Considering $r > 0 \Longrightarrow (\frac{1}{1+r}) < 1$, the second portion of equation's (4) right hand side will become negligible. If we further assume a mathematical specification for the stochastic process governing dividends, $E_t[D_{t+i}]$, it's possible to present today's price (P_t) depending only on parameters (i.e reduced form). The most common choice is the first order auto regressive process. (See for example Phillips et al. (2013) and Evans (1991)). In Section (5.1), the mathematical law used to simulate the fundamental price of the bubbled security and the broad index is identical to the one demonstrated below.

Assuming that the dividends can be represented by a stochastic process such as: $D_t = \mu + D_{t-1} + \epsilon_t$, $\epsilon_t \sim N(o, \sigma^2)$, *the price process will be given by:*

$$P_t = \left(\frac{1+r}{r^2}\right)\mu + \left(\frac{1}{r}\right)D_t \tag{6}$$

[2]In other fields, as for example in policy design, it might be called "sunspot"

3.3 Rational Bubbles

As previously mentioned, the rise of speculative bubbles is consistent with the rational expectations hypothesis. This result is supported by the general idea that any investor is willing to pay a price above the fundamental value, as long the price will get even higher. In the present section this property will be fully explored by reviewing the most common Bubble processes employed in the literature as well as their importance in the simulation procedure of the new anticipation mechanism.

Its possible to write the current price (P_t) as the sum of two different contributions, as such:

$$P_t = F_t + B_t \tag{7}$$

At any given point in time the price encapsulates both fundamentals and an extrinsic variable (i.e Bubble).

Equation (7) constitutes a valid solution to the difference equation given by (3), if the stochastic process driving the bubble component (B_t) respects the following specification:

$$B_{t+1} = (1 + r)B_t + \epsilon \Longrightarrow E_t[B_{t+1}] = (1 + r)B_t \tag{8}$$

The lack of empirical truth regarding the process (8) is very clear. According to Kindleberger & Aliber (2005), the price exuberance will stop at a certain point in time, where the former process continues to explode as time goes to infinity. To overcome such lack of reality, the literature presents alternative bubble specifications as presented next.

3.3.1 Bubbles a la Blanchard (1979)

The base for all exuberance testing across the literature is the stochastic process developed in Blanchard (1979), known by *periodically collapsing bubbles*.

The specification of such process is presented below:

$$B_{t+1} = \begin{cases} \frac{1+r}{\pi} B_t + \epsilon_t, & \text{with probability} \quad \pi \\ \epsilon_t, & \text{with probability} \quad (1-\pi) \end{cases} \tag{9}$$

In contrast with (8) , the bubble component can assume two distinct processes: in the first it can grow at rate $(\frac{1+r}{\pi})$ which is higher than $(1+r)$ but at a certain point in time it can collapse with probability (π) and it will become a white noise process (ϵ_t).

The stochastic process defined by (9) satisfies the condition given by (8) and by consequence is also a solution for equation (3) and (7). The mathematical demonstration of the former statement can be found in Section (9.2.6).

3.3.2 Bubbles a la Evans (1991)

The specification given by equation (9) is the base structure on which the Evans (1991) process was built on, however in the former, it is possible to control **two additional aspects** beyond the probability of a collapse (π):

$$B_{t+1} = \begin{cases} (1+r) \times B_t \times u_{t+1} \ \text{ if } \ B_t \leq \alpha \\ [\delta + \frac{(1+r)}{\pi} \times \theta_{t+1} \times (B_t - \frac{\delta}{(1+r)})] \times u_{t+1} \ \text{ if } \ B_t > \alpha \end{cases} \tag{10}$$

When varying the parameters ($\alpha > 0$) and ($\delta > 0$), one is controlling for the bubbles average length before they collapse and the frequency for which the bubbles erupt, respectively. In both processes, (9) and (10) , the probability of occurring a collapse ($\pi \in [0, 1]$) controls the exuberance scale, however in (10), this parameter's influence acts through a Bernoulli trial (θ_{t+1}) occurring in each period, which can allow the on going bubble ($B_t > \alpha$) to continue even further or collapse the whole process to the restarting value δ. Last but not least, the multiplicative disturbance is given by: $u_t = \exp[y_t - \frac{\tau^2}{2}]$, $y_t \sim N(0, \tau^2)$, $E_[u_{t+1}] = 1$

The stochastic process defined by (10) satisfies the condition given by (8) and consequently is also a solution for equation (3) and (7).

The former process is by far the most influential in bubbles literature and has a major role in the simulation of the new anticipation mechanism as its demonstrated in section (5.1).

3.4 The 'Adaptive Learning' Theory

The most influential literature on price exuberance was built on Rational Expectations, however this method of generating expectations imply some far-fetched assumptions, namely the agent's complete knowledge and understanding of the economy's structure, regardless its high level of complexity.

The 'Adaptive Learning' Theory provides a more realist framework, that enables economists to relax this assumption. The underlying intuition establish economic agents as to that try to include the most recent data in their expectations by constantly updating them as newly data is released. From a modeling perspective this means that agents constantly re-estimate their set of structural parameters instead of performing calibration or simply assume a certain value. To emphasize how the Learning Theory offers a robust a proper theoretical extension, Bray (1982), demonstrates the parameters' asymptotic convergence towards the Rational Expectations equilibrium, as the agents sample size increases to infinity. Establishing a seminal paper for this approach is quite a challenge in it self however the reader can find a complete treatment of this subject in Evans & Honkapohja (2001).

As one might expect, the estimation methods applied in 'Adaptive Learning' extends beyond the standard ordinary least squares. Instead of a full sample approach where parameters are not allowed to change, the estimation is performed through a set of updating equations that are fed when newly data is available, thus originating a new set of point estimates. Recursive estimation implies specific techniques to deal with parameters variation. To illustrate this estimation method I follow Evans & Honkapohja (2001) however a full formalization on the subject is presented in Young (2011).

The procedure starts with obtaining an expectation for a given economic variable (Y_t^e) which depends on a set of other independent variables ($\mathbf{X}_t$) and parameters estimates ($\hat{\theta}_{t-1}$), this translates mathematically into:

$$Y_t^e = \Psi(\mathbf{X}_t, \hat{\theta}_{t-1}) \tag{11}$$

The main difference between R.E and Adaptive learning is on how agents obtain estimates for $\hat{\theta}_{t-1}$. Instead of assuming a certain value for the parameters' set[3], the agents will constantly update their estimates with new data according to the following algorithm:

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \gamma Q(t, \hat{\theta}_{t-1}, \mathbf{X}_t) \tag{12}$$

Where γ is a measure of how the algorithm (12) should respond to new data and Q represents the specification of the updating equation.

It is easy to anticipate the relevance of the 'Learning' approach when market conditions start to reveal the presence of a speculative bubble. Nevertheless the importance of this subject regarding the new anticipation mechanism will be fully discussed in section (4.1).

[3]This technique is called calibration

3.5 CAPM Equilibrium Model

A very important part of the proposed anticipation mechanism relates with the utilization of an asset pricing model to provide a financial meaning to the observed price explosiveness. The Capital Asset Pricing Model (CAPM) presented in Sharpe (1964) constitutes the most sought out equilibrium model in finance. In basic terms this model yields the equilibrium rate of return of a given security under a set of assumptions. The most notorious, specify that all market participants should maximize their reward to volatility ratio, which is the same as stating they build their portfolios as developed in Markowitz (1952). Although this portfolio construction method is the baseline for modern portfolio theory its hardly utilized due to its grotesque number of estimations related to the cross correlation between different assets. The CAPM's main specification is given by:

$$E_t[R_{i,t}]^{\text{CAPM}} = r_f + \beta(E_t[R_{M,t}] - r_f) \tag{13}$$

Where $E_t[R_{i,t}]^{\text{CAPM}}$ represents the stock i equilibrium return at time t; $\beta \equiv \frac{COV(R_{i,t},R_{M,t})}{VAR[R_M]}$ is a measure of risk that shows the contribution of the asset's volatility to the total market volatility at time t; r_f is a risk free rate of return and finally $R_{m,t}$ is the total market return at time t.

The specification (13) possess a serious limitation: the market return is not observable, due to its theoretical nature. In Sharpe (1963) the author proposes an alternative approach to the Markowitz (1952) algorithm which is also a commonly used solution to the CAPM's theoretical Market portfolio. The Single Index Model (SIM) makes uses of the following statistical regression:

$$R_{i,t} - r_f = \alpha + \beta(R_{I,t} - r_f) + \epsilon_t, \qquad e_t \sim N(0, \sigma^2) \tag{14}$$

Where α and β are fixed parameters and $R_{I,t}$ is the return at time t of a benchmark portfolio[4], which can be used to proxy the CAPM's Market portfolio. As one would expect the β is identical to the one in specification (13) and α is the CAPM's abnormal return prediction for the stock i. The main limitation of the SIM relies in its simplicity, because it's a linear regression it should satisfy the underlying assumptions to provide trustworthy results.

3.6 The Backward Supreme Augmented Dickey Fuller Statistic (BSADFS)

A very important part of the proposed anticipation mechanism is how it captures the observed price acceleration. The most effective way to complete this task is using the BSADFS date stamp strategy developed in Phillips et al. (2013). In basic terms this statistic selects the highest value of the standard Augmented Dickey Fuller test in various increasing sub samples. The estimators will identify the fractional beginning of exuberance ($\widehat{r}_e$) by the first period in which the BSADFS is higher than the simulated critical value and its fractional ending ($\widehat{r}_f$) for the first period in which the BSADFS is lower than the simulated critical value. In mathematical terms the estimated fractional begining and termination of a bubble is given by equation (15) and (16), respectively:

$$\widehat{r}_e = \inf_{r_2 \in [0,1]} \{r_2 : BSADFS_{r_2}(r_0) > scv_{r_2}^{\beta_T}\} \tag{15}$$

$$\widehat{r}_f = \inf_{r_2 \in [\widehat{r}_e + \delta \log(T)/T, 1]} \{r_2 : BSADFS_{r_2}(r_0) < scv_{r_2}^{\beta_T}\} \tag{16}$$

Where ($\widehat{r}_e$) and ($\widehat{r}_f$) represent, respectively, the fractional beginning and termination of exuberance. The value given by r_0 represent the percentage of the total sample used to initiate the calculation of the BSADF whereas

[4]The Benchmark is very often a Broad Equity Index like the Nasdaq Composite

r_2 represent the last point of the sub sample in test, given as a percentage of the total sample size (T). The BSADF statistic performs a comparison between backward expanding sub samples in which the starting point varies from 0 to $(r_2 - r_0)$. One can obtain the exact estimated period for the bubble outbreak and its respective termination by calculating $\widehat{r}_e \times T$ and $\widehat{r}_f \times T$. Finally $scv_{r_2}^{\beta_T}$ represents a simulated critical value which depends on given significance level (β_T) and total sample size (T).

4 A New Bubble Anticipation Mechanism

Now that the reader is familiarized with the concepts presented above, I will demonstrate how they can compose a new system capable of anticipating a speculative bubble. For ease of reference the new anticipation mechanism will be called MMBA (Moreira Martins Bubble Anticipator).

4.1 Designing the MMBA Mechanism

The novelty underlying this thesis is forecasting price exuberance through the combination of statistical analysis and asset pricing equilibrium. The objective behind the mechanism, is generating a **one step ahead probability** forecast towards the existence of a speculative bubble, using a *dynamic probit model*. The adequacy of a binary response model to a problem such as this is justified by our inability to **directly observe an ongoing bubble**. In fact we can only observe a sudden phenomenon of great acceleration in the security price as showed in Franses (2013). This kind of behavior is easily captured through the BSADF statistic as demonstrated in Phillips et al. (2013). The main challenge is **differentiating between explosive unobservable fundamentals and a speculative bubble** Hamilton (1986). The new MMBA mechanism tries to tackle this gap by using the following approach: Lets apply an asset pricing equilibrium model and a non stationarity statistic to the same data stretch and collect their predictions under the same econometric specification. This not only allows the investor to estimate the specific contributions of the former components to the bubble structure, but also constitutes a powerful tool in the ex ante optimization of the portfolio. In sum, the MMBA will grant a robust method to perform a "bubble surf" strategy highly adopted in this type of events as demonstrated in Brunnermeier & Abreu (2003).

4.2 The MMBA Procedure

The main econometric model is given by:

$$\begin{cases} \text{bubble}^*_t = \beta_1 + \beta_2 \Upsilon^*_{t-1} + \beta_3 D^{\text{BSADFS}}_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(\mu, \sigma^2) \\ D^{\text{BSADFS}}_t = I(\text{bubble}^*_t > 0) \end{cases} \tag{17}$$

Where the latent variable (bubble^*_t) is a function of the **current abnormal return, of a given asset pricing model** (Υ^*_{t-1}) and a dummy variable accounting for today's price explosiveness:

$$D^{\text{BSADFS}}_{t-1} = \begin{cases} 1 \text{ if } \text{BSADFS}_{t-1} > s.c.v^{\text{BSADFS}} \\ 0 \text{ if } \text{BSADFS}_{t-1} \leq s.c.v^{\text{BSADFS}} \end{cases}$$

Where $s.c.v^{\text{BSADFS}}$ represents the simulated critical value for a given significance level [5] and ϵ_t is a Normal distributed error with mean μ and variance σ^2. The indicator function ($I(.)$) assumes the value one if the unobserved bubble exists (i.e, $\text{bubble}^*_t > 0$) and the value zero if the bubble does not exist (i.e, $\text{bubble}^*_t \leq 0$). The intuition is similar to the one in equation (7), if bubbles assume a null or negative value, which in practice means they don't exist ($B_t = 0$), then the asset price is equal to the respective fundamental price ($P_t = F_t$).

[5] For each significance level, 1%, 5% and 10% was performed a simulation of 5000 replicas of a CUMSUM process with total sample size of 1000.

Intuitively, the specification given in (17) answers the following question: *What is the probability of tomorrow's price containing a bubble, given today's abnormal return and the value of the BSADF statistic ?.*

Defining $\mathbf{\Omega_{t-1}} \equiv (1, \Upsilon^*_{t-1}, D^{BSADFS}_{t-1})$, $\beta' \equiv (\beta_1, \beta_2, \beta_3)$ *and assuming Normal distributed disturbances,* $\epsilon_t \sim N(\mu, \sigma^2)$ *, the conditional probability of occurring a bubble one period ahead is given by:* $P[\widehat{bubble_t} \mid \mathbf{\Omega_{t-1}}] = \Phi(\beta\Omega_{t-1})$*, where* $\Phi(.) \equiv$*Normal Cumulative Distribution Function.*

The estimation procedure of specification (17) is totally identical to the *standard probit model.* Nevertheless, the parameter estimation as well as the assymptotic properties of *dynamic binary choice models* can be validated in de Jong & Woutersen (2011).

The MMBA initiates with the estimation of its two main components: the abnormal returns ($\widehat{\Upsilon}_{t-1}$) and the BSADFS. The first component is obtained through the difference between the investor's subjective expectation for the asset return and the equilibrium return given by an asset pricing model, like the CAPM Sharpe (1964)[6]. The second component provides the exuberance estimators ($\widehat{r}_e$ and $\widehat{r}_e$) that will generate a dummy variable that proxies a speculative bubble (D^{BASDFS}_{t-1}). When this two components are plugged in the specification (17), one obtains three different point estimates: the regression intercept ($\widehat{\beta}_1$), the contribution of the current abnormal returns to the probability of exuberance (β_2) and the contribution of current price acceleration to the probability of exuberance (β_3). Using the former estimates the new procedure will yield a probability towards the existence of a speculative bubble one period ahead from which the estimation took place:

$$P_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}] = \Phi[\hat{\beta}'_{t-1}\mathbf{\Omega_{t-1}}] \tag{18}$$

The former expression contains an abuse of notation in order to provide a

[6]One can use several other asset pricing models for this purpose

higher intuitiveness to the estimate probability. The correct notation should be $P_{t-1}[D_t^{\widehat{\text{BSADFS}}} = 1|\Omega_{t-1}]$ and not has presented in equation (18).

The recursive nature of the MMBA implies the re-estimation of (17) each time new data is available, in order to provide a probability that incorporates all available information, which not only increases the model's accuracy but is the correct approach to capture changing parameters. A highly detailed simulation of the MMBA is presented from section (5.1) to section (5.4).

4.3 Contribution to the literature

The former mechanism tries to tackle two persistent gaps in the speculative bubbles literature. The first one is concern with the incapability of existing econometric methods to differentiate between imperceptible fundamentals and price exuberance, as extensively discussed in Hamilton (1986) and in Gurkaynak (2005). The new mechanism links non stationarity (BSADFS) and the abnormal returns generated with an asset pricing equilibrium model prediction (Υ^*_{t-1}). By doing so, the model can now judge the explosiveness in the observed price, using the same fundamentals that generate the equilibrium rate of return, thus providing a much clearer conclusion towards the existence of a bubble. In basic terms, one can state that what BSADFS once identified as simply non stationary price behavior, has now a financial meaning through the asset pricing model.

The second gap is the ability to capture what Shiller (2000) stated as a "feedback loop". According to this article, an important part of a speculative bubble rests in its social component, that generates a dynamic behavior that feeds the bubble until its inevitable collapse. The econometric specification given in (17) can capture this process through the lagged dependent variable, because the current value of the variable helps determinate the value of that same variable in the future.

Finally an essential part of the MMBA involves generating expectations using Learning instead of R.E, which provides a substantial update to the most influential literature.

5 Simulation Procedure

The following sections explain in great detail the designed simulation procedure to test the mechanism effectiveness in anticipating price exuberance. The presented and following sections are totally based on Matlab ® programming. From Section (5.1) to (5.4) the results of a single replica are meticulously presented. In Section (5.5) I comment on the one hundred replicas performed to the MMBA procedure, in order to derive its asymptotic properties. Table 1, presented next, illustrates the standard choice of parameters for a single replica and for the monte carlo simulations in section (5.5).

Table 1: Standard Parameters

Parameter Description	Standard Values			
Sample Size	$T = 1000$			
MMBA Initial Sample	$20\% \times T = 200$			
Security Price	$P_0^S = F_0^S = 100$	$D_0^S = 1.3$	$\mu = 0.0000746$	$\sigma^2_{D^S} = 0.0003148$
Index Price	$P_0^I = F_0^I = 1000$	$D_0^I = 10$	$\mu = 0.0373$	$\sigma^2_{D^I} = 0.1574$
Evans (1991) D.G.P	$B_0 = 0$, $\alpha = 1$, $\delta = 0.5$, $\pi = 0.85$			
Risk Free Rate	$r = 5\%$			
BSADF Significance Level	$\beta = 5\%$			
Constant Gains	$\gamma = 0.2$			
Asset Pricing Model Contribution	$\widehat{\beta}_2 \neq 0$			
P-Value Validity	Yes			

5.1 Generating Price Exuberance

The whole procedure initiates by simulating two structurally different time series, one for the security and other for the benchmark index. First the fundamental component alone of the former is simulated according to the law given by (6). To obtain the observed market price, given by equation (7), we sum it to the bubble process represented in (10). The discount factor was deemed constant and was estimated using 6 Months Treasury Bills monthly data from 1960 to 2014. The simulation total sample size is eight hundred (T=800), however the BSADF statistic consumes a variable mandatory initial sample of $(r_0 * T - 1)$, where $r_0 = 0.01 + \frac{1.8}{\sqrt{T}}$. As the simulations in Section (5.5) indicates, the MMBA also performs better with an initial sample, for which the best results occur with $20\% \times T$. Subtracting both of this from the total sample size (T) will yield a final sample size of 583 showed in all figures.

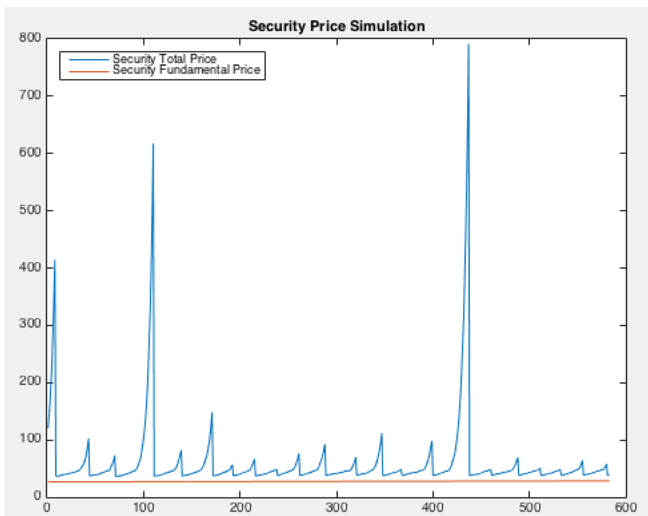

Figure 1: Simulated Security Price according to (6), (7) and (10), T=583

To emphasize the graphical effect of price exuberance both graphs also exhibit the discounted dividend stream (fundamental price). One can certainly check the existence of three major peaks to be anticipated. One in the beginning of the sample, another one around the 100th period and a final one around the 400th period. The following estimated variables should exhibit a very specific behavior around this periods which will help the MMBA predict one step ahead.

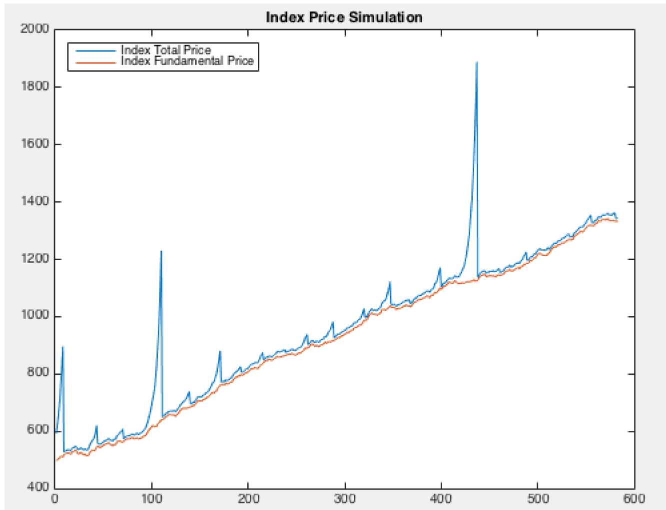

Figure 2: Simulated Benchmark Price according to (6), (7) and (10), T=583

5.2 'Adaptive Learning' & Abnormal Returns

The next step is based on the recursive estimation of the security's abnormal return. This value can be calculated by comparing a subjective prediction of the security's return with the equilibrium return generated by a given asset pricing equilibrium model. Due to its generalized influence the following simulation will make use of the CAPM equilibrium return. However one can use any equilibrium model deemed fit to the analyzed security. The subjective prediction of return will result from the adaptive learning theory, as discussed in section (3.4).

The choice of the learning rule is strongly related with the type of security analyzed. For general purposes the learning process will be based on the mean rate of return of the security:

$$R_{t+1} = R_t + \epsilon_{t+1}, \;\; \epsilon_{t+1} \sim N(0, \sigma^2) \tag{19}$$

Where R_{t+1} is the security return at period t+1 and can be obtained by:

$R_{t+1} = \frac{P_{t+1} - P_t}{P_t}$. The ϵ_{t+1} represents a white noise disturbance.

The learning rule (19) will be updated according to the following recursive algorithm:

$$\bar{R}_t = \bar{R}_{t-1} + \gamma[R_t - \bar{R}_{t-1}] \tag{20}$$

$\bar{R}_t \equiv$ Asset Mean Return until time t
$\gamma \equiv$ Constant Gain Parameter
It's easy to verify that the algorithm (20) represents a particular case of the one represented by (12).

In each period the abnormal return is estimated using the following equation:

$$\widehat{\Upsilon}_t^* = \bar{R}_t - E[R_t]^{\text{CAPM}} \tag{21}$$

In intuitive terms, the abnormal return, represented in figure (5) is the difference between the investor's expectations given in figure (4) and the asset fundamentals represented by the CAPM equilibrium prediction which can be observed in figure (3). Each of the former variables will be updated each time new data is available.

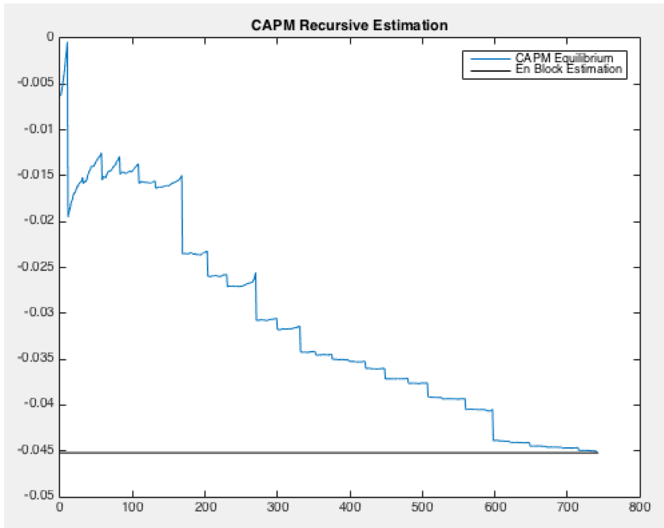

Figure 3: Security CAPM Equilibrium Return ($E[R_t^{\text{CAPM}}]$), T=583

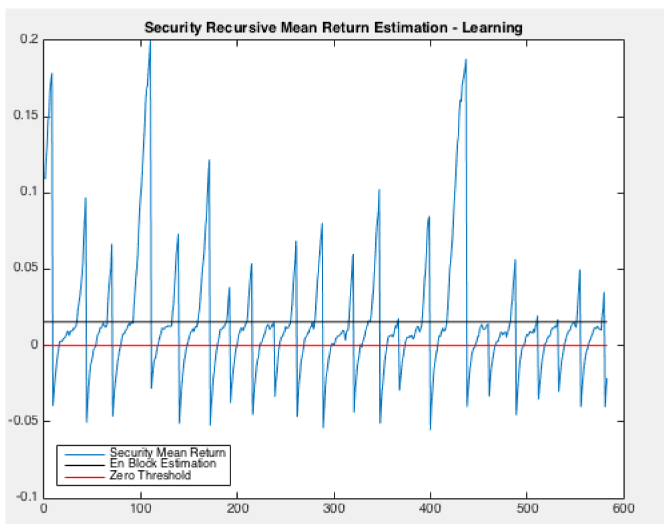

Figure 4: Security Mean Return ($\bar{R}_t$) - Learning Rule, T=583

The huge peaks observed in the abnormal returns are a direct result from the learning rule estimation approach. **Estimating expectations with constant gains is essential to the MMBA**, because it's an effective way to incorporate the significant structural breaks induced by price exuberance

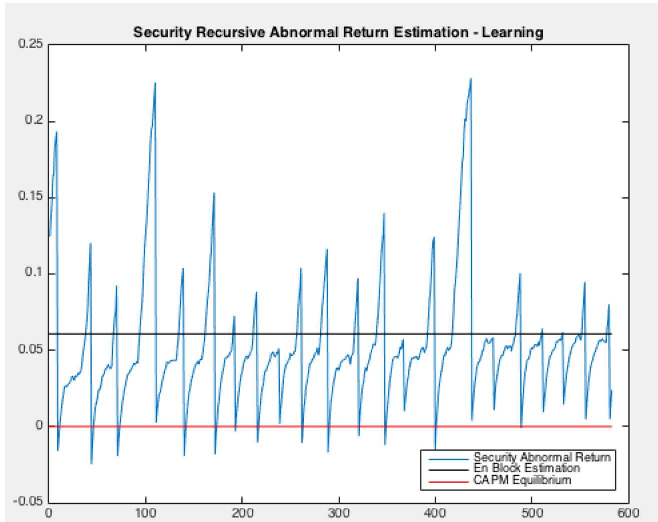

Figure 5: Security Abnormal Return ($\widehat{\Upsilon_t}$), T=583

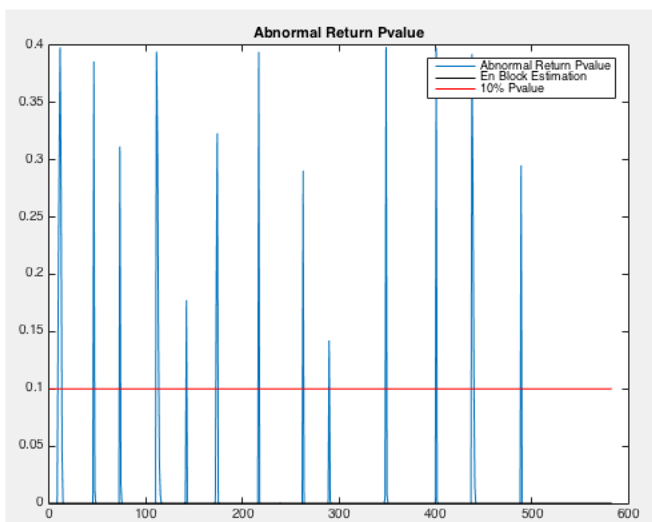

Figure 6: Security Abnormal Return P-Value, T=583

in the equilibrium variables. Considering an econometric perspective, the abnormal returns benefit from a robust statistical significance in any of the problematic periods, demonstrating the estimation effectiveness in capturing structure changes induced by exuberance.

5.3 Measuring explosiveness with BSADFS

The Next step will measure the current price (P_t) acceleration with the BSADFS. As its demonstrated in section (5.5), feeding this information to the MMBA will significantly improve predictability. Once again, figure 7 exhibits maximum values in the quoted periods, which largely surpass the 1% critical value, implying an extreme price acceleration similar to the ones observed in real world events.

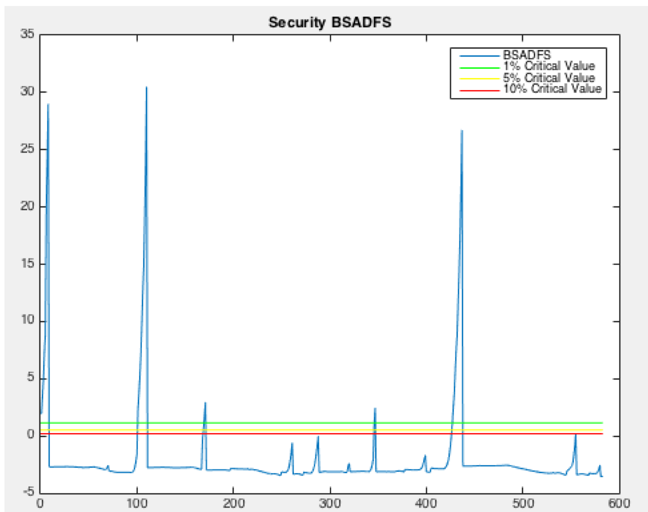

Figure 7: BSADFS - Security Price, T=583

5.4 Dynamic Probit Estimation

Arriving at the procedure's last stage, the data signals obtained in sections 5.2 and 5.3 are fed into the MMBA specification (17) detailed in section (4.2). This exercise will yield three different point estimates beside the one step ahead probability: the regression intercept ($\widehat{\beta}_1$) presented in figure (8), the abnormal return contribution ($\widehat{\beta}_2$) which can be observed in figure (10) and finally the BSADFS contribution ($\widehat{\beta}_3$) given in figure (12). In each period a standard T test is perform for each of the former estimates, the associated p-values can be observed in figures (9), (11) and (13).

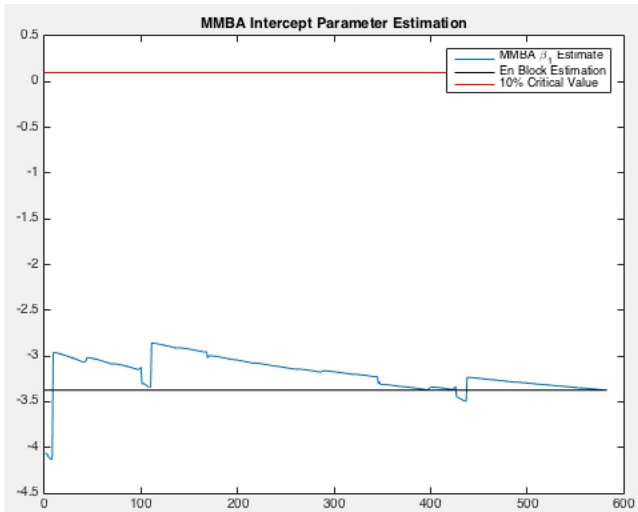

Figure 8: MMBA Intercept Estimation ($\widehat{\beta_1}$), T=583

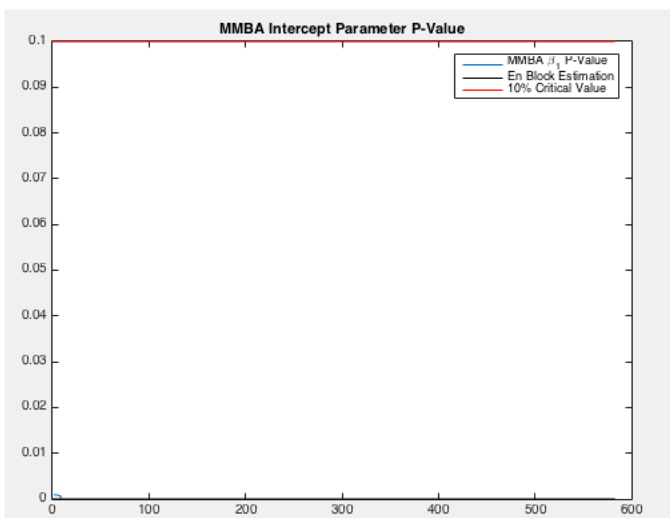

Figure 9: MMBA Intercept T-test P-Value, T=583

Once again, the parameter's sharp increase in the referenced periods is a

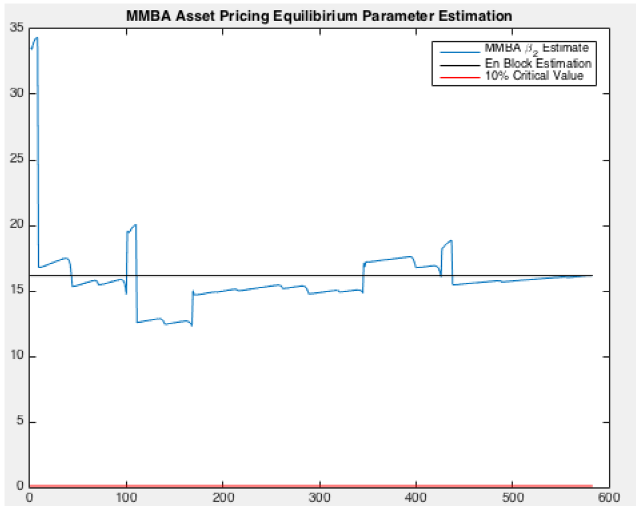

Figure 10: MMBA Abnormal Return Parameter Estimation ($\widehat{\beta_2}$), T=583

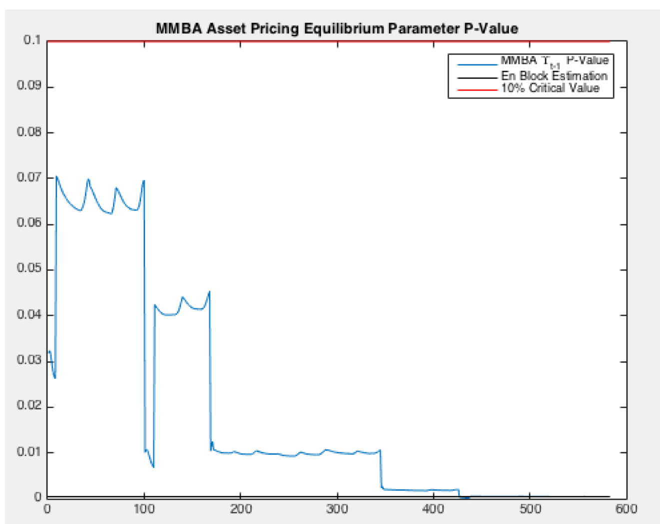

Figure 11: MMBA Abnormal Return T-test P-Value, T=583

direct result from bubble's occurrence, which also leads to salient increases in their statistical significance across all parameters.

In each period the MMBA yields a one step ahead probability, however this result is only trustworthy if and only if all parameters are statistical

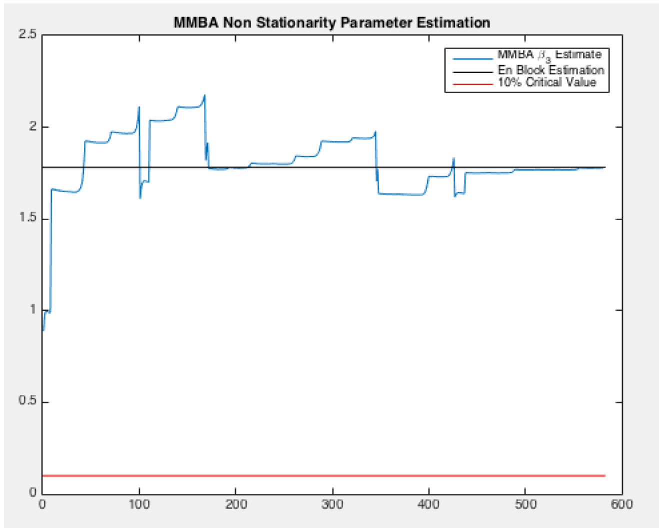

Figure 12: MMBA BSADF Statistic Parameter Estimation ($\widehat{\beta_3}$), T=583

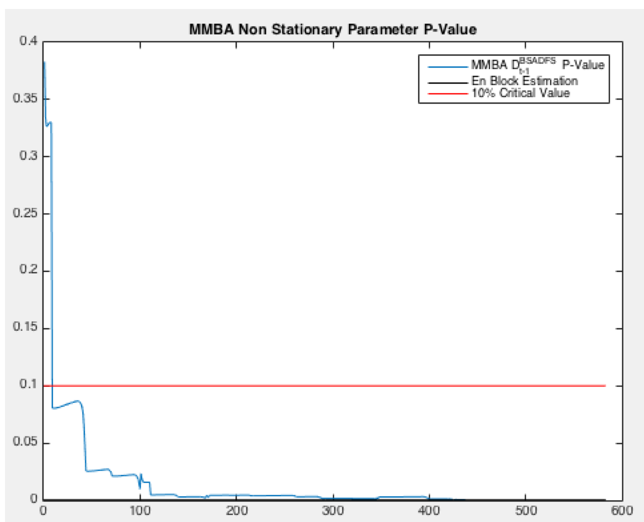

Figure 13: MMBA BSADF Statistic T-test P-Value, T=583

significant, because it's not rational for any investor to perform changes in his portfolio if his/her decision model is lacking good statistical properties. For this reason figure 8 presents two types of probability considered jointly: a simple one given by $P_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}]$ and validation measure represented

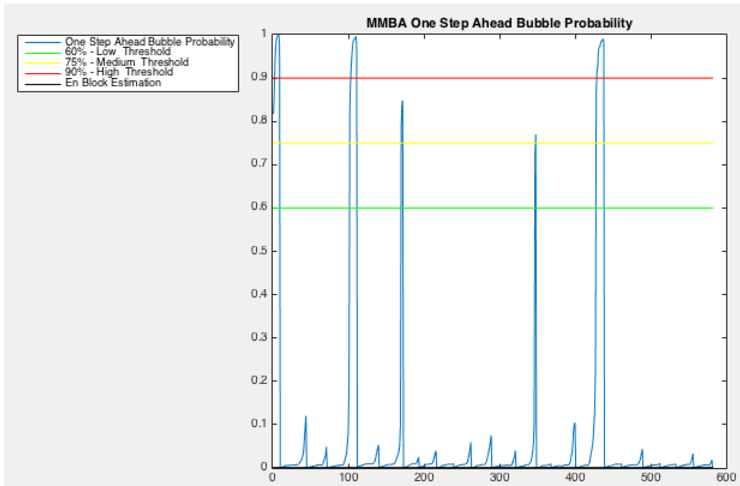

Figure 14: MMBA One Step Ahead Probability, T=583 ($P_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}]$)

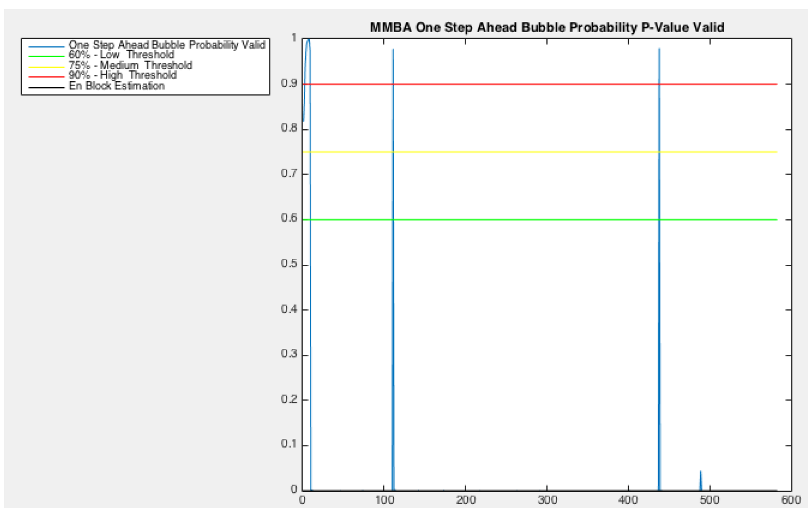

Figure 15: MMBA One Step Ahead Validated Probability, T=583 ($P^*_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}]$)

by:

$$P^*_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}] = \begin{cases} P_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}] & \text{if} \quad \text{Validity}_{t-1} = 1 \\ 0 & \text{if} \quad \text{Validity}_{t-1} = 0 \end{cases} \tag{22}$$

Where the variable Validity_{t-1} is defined as:

$$\text{Validity}_{t-1} = \begin{cases} 1 & \text{if} \quad \text{Pvalue}\left(t_{\widehat{\beta}_1}, t_{\widehat{\beta}_2}, t_{\widehat{\beta}_3}\right) \leq 10\% \\ 0 & \text{if} \quad \text{Pvalue}\left(t_{\widehat{\beta}_1}, t_{\widehat{\beta}_2}, t_{\widehat{\beta}_3}\right) > 10\% \end{cases} \tag{23}$$

This probability, $\widehat{P}^*_{t-1}$, will be the one considered when testing for anticipation capability. A simple observation of the figure's right hand side reveals that the MMBA procedure was able to yield a high probability one period before the referenced peaks. Taking in consideration that its a validated probability one can state that **the MMBA has successfully anticipated the three bubbles generated in this sample simulation**.

The difference between both probabilities, $\widehat{P}^*_{t-1}$ and $\widehat{P}_{t-1}$ is being cause by the abnormal return P-Value since all the other parameters are statistical significant across all periods. This fact leads to a very important conclusion: *If the observed abnormal return does not have statistical significance then it's impossible for the MMBA to fully distinguish and purge the fundamentals from the observed price.* To fully understand this conclusion, one needs to recall that the abnormal return results from the difference between the fundamentals given by the asset pricing model and the investor's subjective expectations (see Section 5.2). A lack of statistical significance implies that at least one of the former components is not capturing the observed data, leaving no choice but to ignore the one step ahead probability.

5.5 Monte Carlo Simulations

In this sub section, a monte carlo exercise is performed to infer about the MMBA statistical properties. These were performed with a standard set of parameters illustrated in table 1, which only suffer modifications when the specific simulation requires. The standard set was in accordance to Evans (1991) and allows the reader to do the necessary comparisons. With the exception of table 2 that describes the variables in use, all other tables contain three statistical measures: Mean ($\bar{X}$), Standard Deviation ($\Sigma(X)$) and Median ($Med(X)$). Furthermore they are divided into 5 major areas: General descriptive statistics, whose purpose is providing a Birdseye view for each simulation; All Bubble Statistics which provides the MMBA's behavior for all bubbles in the sample; End Sample Bubbles Statistics whose purpose is identical to the former one but only for the last bubble in the sample; All bubbles probability distribution that illustrates how the MMBA probabilities perform before a peak for all the bubbles in sample and finally the End Sample probability distribution which is identical but only for the last bubble in the sample. Along the present section the reference probability for measuring MMBA's anticipation power is the 90% threshold. All tables are contained in the appendix section and each one of them contain the results of one hundred repetitions ($N = 100$), except table 1 presented in section (5).

An important aspect of any econometric procedure is how it reacts to different sample sizes, table 3 provides evidence of how the MMBA predictive power behaves to various samples sizes. It is an obvious conclusion that the new procedure will perform better with larger sample sizes, however this result is not only due to mechanism characteristics but it is also a consequence from the data generating process, which tends to generates a low number of bubbles in small samples ($T = 200$, $T = 400$). This can be seen by the quantity $\#(B_{1,..,T} = 1)$, presenting a mean value of 1.5 bubbles and 5 bubbles for the smallest and largest sample, respectively. This means the most accurate simulations are going to be the ones with the highest sample size, which is $T = 1000$. In fact, one can also observe a huge difference in valid probabilities when comparing sample sizes of $T = 200$ against $T = 1000$, which is 8.20% and 84.95 %, respectively. The MMBA's need for a large sample size does not constitute an issue because the type of financial data necessary to run the procedure can be easily obtained[7]. The new mechanism also reveals a desired learning dynamic, resulting in higher predictive power towards the last bubble in the sample. The simulations showed a mean value of 77.88% for the last bubble against a mean value of 66.44% for all bubbles in the sample, considering a one step ahead probability higher than 60%. Another important consequence of a larger sample size, is the mechanism's ability to present significant probabilities until the fifth period before the bubble peak, which helps execute a better surfing strategy. In one hundred simulations with the largest sample size ($T = 1000$), more than half can yield a probability higher than 60% five periods before the bubble peak, but when one considers the same probability with the lowest sample size ($T = 200$), the former value remains at 24.35%. Last but not least it's possible to testify the one step ahead probability convergence towards the mean value of 70% as the sample size increases.

A closely related subject is the MMBA initial sample presented in table 4. This table exposes how the mechanism react when different sample sizes are used to calculate the **first** one step ahead probability ($\widehat{P}_{t-1}$). The most important result of this simulation is the positive relation between anticipation power and initial sample size, which is in line with the learning dynamic formerly observed. Ilustrating with an example, if one considers a total sample size of one thousand (T=1000), the MMBA will predict better if one utilizes a sample of 200 ($20\% \times T$) to generate the first one step ahead probability ($\widehat{P}_{t-1}$) than if one utilizes a sample of only 100 ($10\% \times T$) for the same effect. The results revealed that the MMBA could provide a one step ahead probability higher than 90% ($P^*_{t-1}[B_t = 1|\Omega_{t-1}] \geq 90\%$) for 41% of all the bubbles and 57% for the end bubbles, when the mechanism does not consume any data to provide the first one step ahead probability, meaning ($0\% \times T$). When using 20% of the total sample ($20\% \times T$) the mechanism

[7]The Bloomberg® Professional Service is a good example of easy one can obtain financial data

could anticipate 54% of all bubbles and 61 % of the end sample bubbles with the same level of probability, which represents the best result. The MMBA yields a higher percentage of valid predictions when 40% of total sample size is consumed, but the anticipation power as a whole is much smaller than the former ones.

The importance of how the abnormal returns are estimated is absolutely crucial for the mechanism's effectiveness. As a consequence its important to observe how the MMBA reacts to different constant gains values (γ). Focusing on table 5 one perceives some sensitivity of the prediction power towards this parameter, especially for the highest value considered ($\gamma = 0.4$). In this last setting the MMBA can provide a one step ahead probability higher than 90% ($P^*_{t-1}[B_t = 1|\Omega_{t-1}] \geq 90\%$) for approximately 47% of all generated bubbles and 43% of the 100 end sample bubbles whereas if one considered the standard value ($\gamma = 0.2$)[8] the MMBA is able to capture approximately 55% of all bubbles and 62% of the 100 end sample bubbles with the same level of probability. This is not a surprising result because this parameter is able to control how the investor perceives structural change in his/her learning rule. If this value is too high then it is **his constantly overestimating changes, which ultimately diminishes predicting power.**

The most important result in the whole set is given in table 6, because it emphasizes how important the asset pricing model prediction is for a robust anticipation. At first glance both settings look very similar to each other, however an indepth observation reveals a very interesting pattern. Without an equilibrium model in its specification the MMBA can only provide a one step ahead probability higher than 90 %, for 17% of all bubbles and 10% of the end sample bubbles. In fact this means that the mechanism looses its learning ability by delivering worst results in the last portion of the sample. When considering a five steps ahead probability, the MMBA without fundamentals cannot predict with a probability higher than 90% more than 5% of the

[8]The simulations results considering $\gamma = 0.1$ and $\gamma = 0.2$ are quite similar

end sample bubbles whereas for the same probability threshold but with fundamentals, the MMBA can predict 45% of the end sample bubbles, which represents a 900% improvement. The main conclusion is: **The MMBA presents a higher prediction power when the asset pricing model prediction is included in its specification.** This result has never been found in the literature.

In this final stage it's important to study the MMBA's predictive power viewing different types of bubbles. In line with this reasoning, tables 8, 9 and 10 outline the impact by significant variations in the three DGP parameters presented in section (3.3.2). In table 8 its possible to perceive what happens to the MMBA's predictive power when the average duration of exuberance is modified. One can perceive some significant differences between the simulations. First there is a sharp decrease in the number of valid probabilities when the average duration of each bubble is high ($\alpha = 10$). It's not an intuitive result, when this setting presents the highest percentages of end bubble predictability, yielding a 90 % five steps ahead probability for exactly 65% of the 100 end sample bubbles, representing 30% more than the standard duration ($\alpha = 1$) and 70% more than the lowest duration setting ($\alpha = 0.1$). In line with this result, the MMBA probability distributions present a mean value of 90% and very small standard deviations when compared with the lowest duration settings. This value also tends to increase as the step ahead gets larger. One can then conclude that the MMBA **will perform better with high duration bubbles**. In section 6, this can be observed at the large length of the "Technological Bubble". The small length settings ($\alpha = 0.1$ and $\alpha = 1$) yield similar results for all bubbles in the sample, however there are sharp differences regarding the prediction of the last bubbles in the sample. Considering the standard bubble length ($\alpha = 1$) the MMBA can deliver a one step probability higher than 90% for 62% of the end sample bubbles.

Equally important is how sensitive the MMBA is to different bubble scales. The statistics are presented in table 10. By looking at the results the results is possible to verify how insensitive the MMBA is to small scale bubbles, by the dramatically decline in its prediction power. In the smallest scale setting ($\pi = 0.5$) the mechanism can only predict 1% of all the bubbles in sample and 4% of the end sample bubbles, which invalidates the mechanism's capability to detect small bubbles. The most obvious conclusion **is the positive relation between anticipation power and bubble scale**. In the larger scale setting ($\pi = 0.99$) the MMBA yields the best results of all the performed simulations by anticipating 83% of all bubbles in sample for all probabilities thresholds and anticipation periods. The former results are quite intuitive since the lower the collapse probability $(1 - \pi)$ gets the more closer the DGP gets to an exponential process where collapses cease to exist. Finally one should note how the MMBA executed a perfect learning dynamic in the highest bubble scale ($\pi = 0.99$), predicting 100% of the end sample bubbles for the five periods ahead with the highest probability threshold.

Last but not least it's important to understand how the procedure responds to different bubble eruption frequencies. Table 9 shows a weak performance if there is a large number of bubble eruptions, which might seem contradictory to the learning dynamic formerly described but one should consider the following fact: In Evans (1991) it's emphasized how the D.G.P tends to generate a higher number of smaller scale bubbles when δ is larger than α, which is the case. As demonstrated before the MMBA can only predict bubbles of a certain scale hence it will perform badly for this combination of parameters. Considering the case where the number of bubble eruptions is smaller, the D.G.P will have an opposite behavior, because $\delta < \alpha$ Evans (1991). In this situation the new procedure delivers a proper anticipation power, which leads to the following conclusion: **The MMBA learning dynamic is more responsive to a single large scale bubble than to several small scale bubbles**.

6 Application to Real Data

Having shown the mechanism's capability with simulated data, the following question pops up: *Can the MMBA anticipate real world bubbles ?*

To answer this question, we will apply it to roughly thirty years of Apple and NASDAQ Composite monthly data and test if it can anticipate the "dot-com" bubble by providing a significant one step ahead probability towards the bubble peak.

Figures 16 and 17 shows the price plot for the Apple Stock Price and Nasdaq Composite Index, respectively. The exuberance episode in reference takes place around the year 2000.

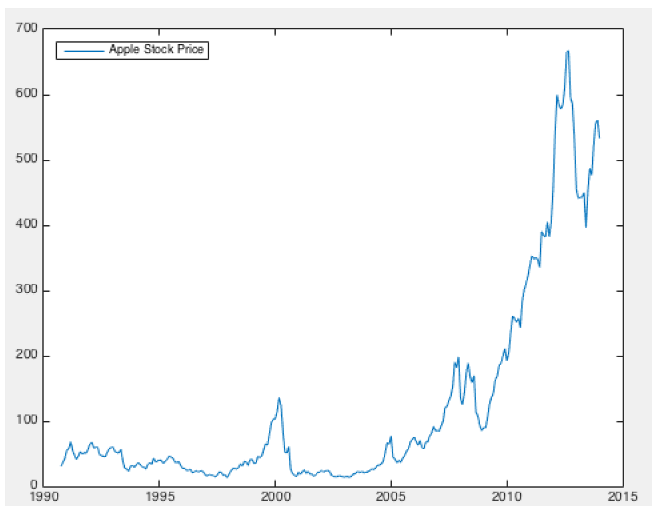

Figure 16: Apple Stock Price from 1990 to 2014

One can easily understand the exacerbated acceleration each security takes in the quoted period. If the investor expects the price to go higher, he will want to take a long position in the apple stock as soon as possible and close that position just before the price starts to collapse and return to its fair value, which means he surfs the bubble as in Brunnermeier & Abreu (2003). In line with section 4.1, the investor starts by building his expectations using

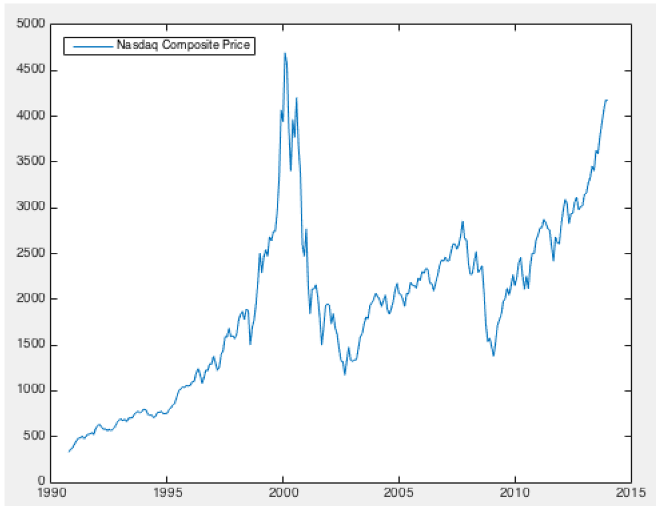

Figure 17: Nasdaq Composite Price from 1990 to 2014

as learning rule the apple's mean return[9], which can be observed in figure 18. These expectations are then compared with the recursive estimation of the CAPM presented in figure 19 , thus yielding the expected abnormal return for the apple stock illustrated in figure 20.

The results are quite similar to the simulations performed in section 5, around the year 2000 one can observe sharp increases in all MMBA parameters and respective statistical significance, emphasizing the idea of a robust estimation mechanism supporting the one step ahead probability. This estimations can be observed in figures 22, 23,24, 25, 26 and 27. The observed level of noise in the Apple and NASDAQ data is much higher than the simulated price in section (5.1), revealing a desired characteristic in the MMBA mechanism, which is the **mechanism's robustness to data variance**. It is worth noting the evolution of the abnormal return and its statistical significance in the referenced period, because it's a revealing indicator of the proper estimation of both fundamentals and subjective rate of return.

[9]As stated before the investor may choose any learning rule to calculate his/her expectations

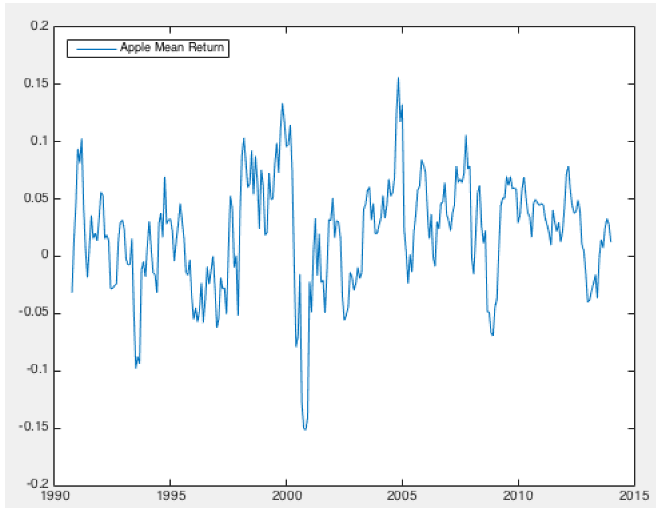

Figure 18: Apple's Mean Return ($\bar{R}_t$), from 1990 to 2014

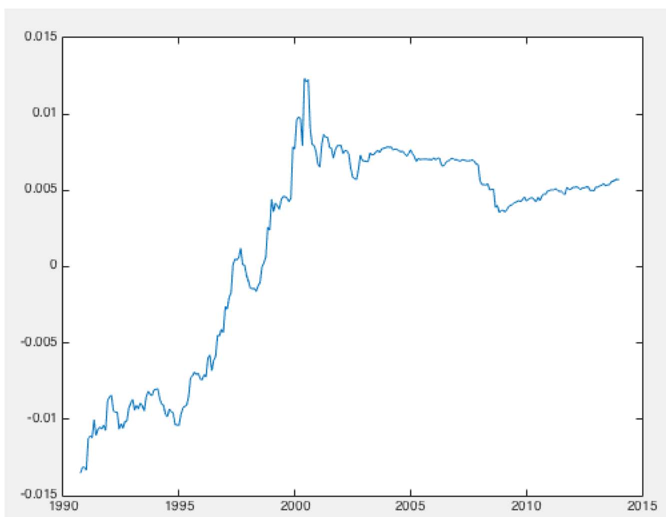

Figure 19: Apple's CAPM Return ($E[R_t]^{\text{CAPM}}$) , from 1990 to 2014

At the validated one step ahead probability, illustrated by figure 29, one perceives that the MMBA mechanism was able to perform a complete anticipation of the "technological bubble", yielding probabilities higher than 85% five months before the bubble peak and a probability higher than 90% one

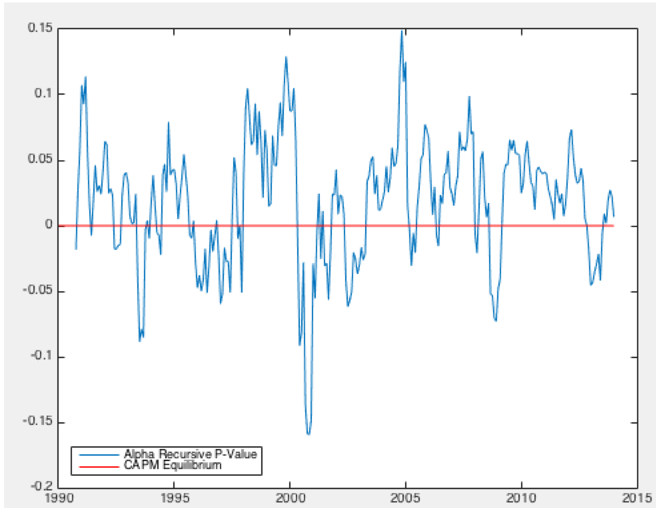

Figure 20: Apple's Abnormal Return ($\widehat{\Upsilon_t}$), from 1990 to 2014

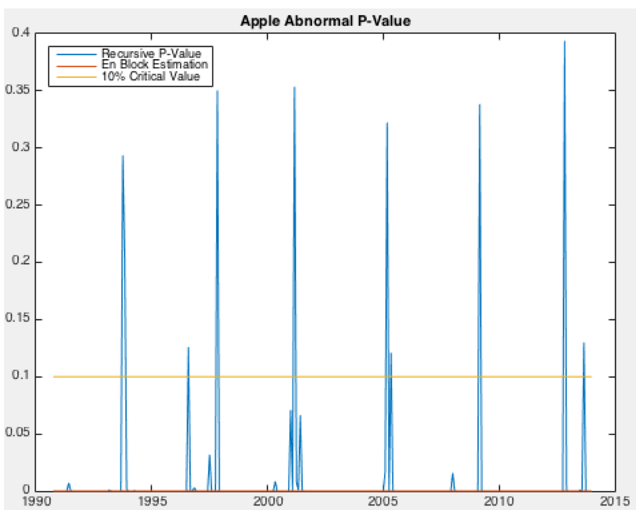

Figure 21: Apple's Abnormal Return T-Statistic P-Value from 1990 to 2014

month before the bubble peak.

Considering the former results one perceives that the MMBA could provide an enormous advantage for two different perspectives operating in the same market: the investors which are interest in maximizing the returns from their portfolio and market regulators[10] whose job is maintain market efficiency. Looking at the specific case of the "Tech Bubble" if an investor would made his/her portfolio adjustments according to the MMBA he/she would have made an enormous profit due to the mechanism high predictive power. In practical terms the investor would have to proceed in the following way: He/She runs the MMBA for the Apple stock, if the result was a one step ahead probability which surpasses a high threshold, for example 80%, then he/she would have a strong incentive to increase the relative weight of the apple stock in the portfolio, which one month from that time would generate a high rate of return. The investor could repeat the strategy until he/she got a clear signal that exuberance has stopped. In a regulator's perspective it's quite reasonable to assume most investors will try to adopt this type of strategy which would have justify a temporary suspension towards the apple stock to avoid a market crash.

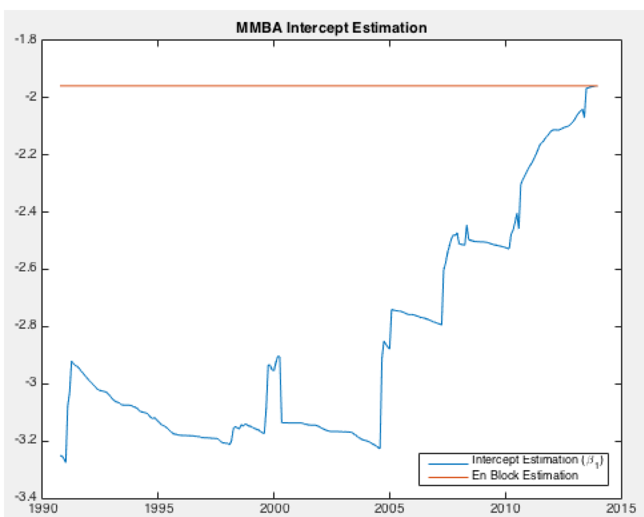

Figure 22: Apple's MMBA Intercept ($\widehat{\beta_1}$), from 1990 to 2014

[10]One can consider the Portuguese market authority CMVM

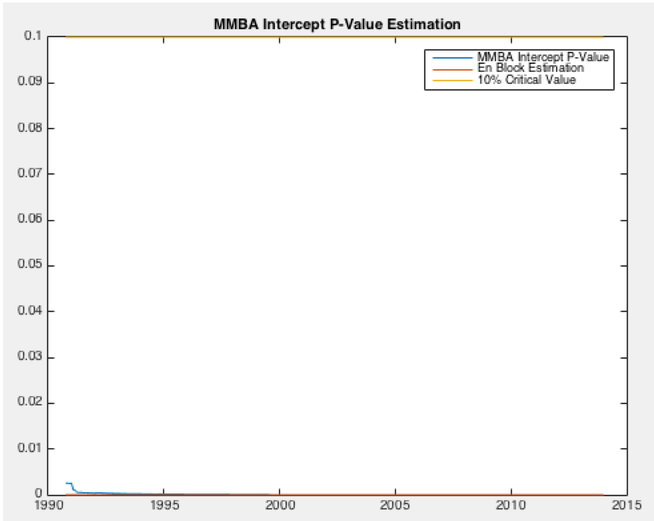

Figure 23: Apple's MMBA Intercept T-Statistic P-Value, from 1990 to 2014

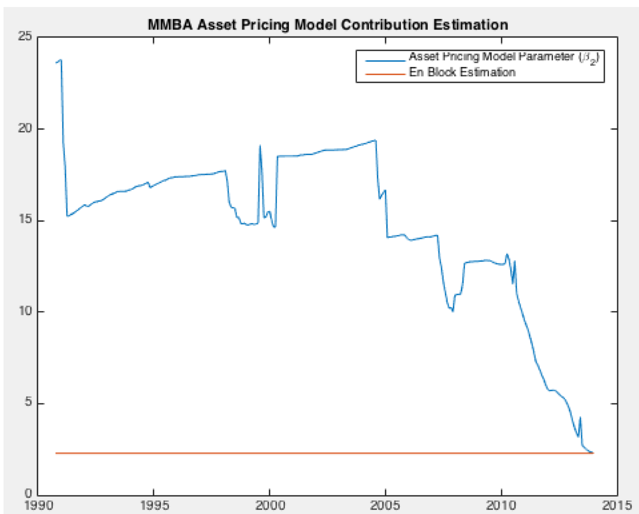

Figure 24: Apple's MMBA Asset Pricing Model Parameter ($\widehat{\beta_2}$), from 1990 to 2014

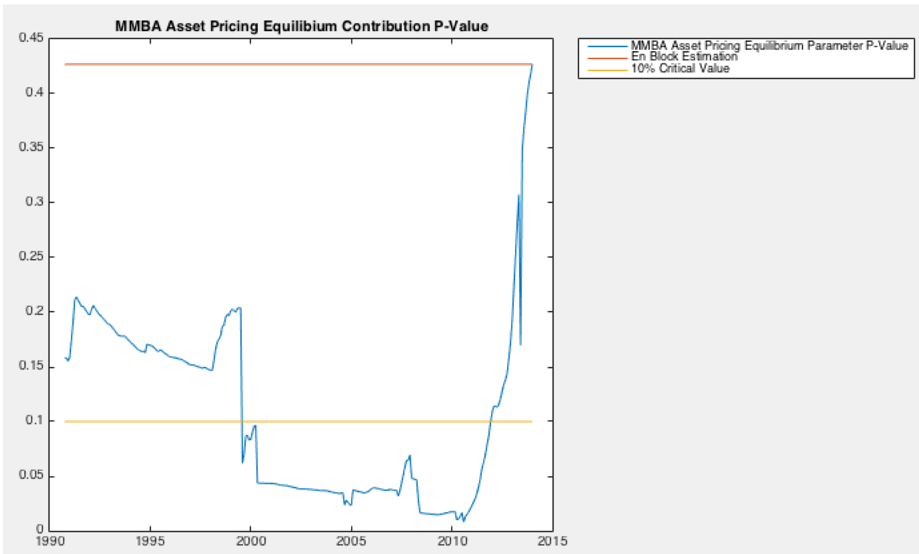

Figure 25: Apple's MMBA Asset Pricing Model Parameter T-Statistic P-Value, from 1990 to 2014

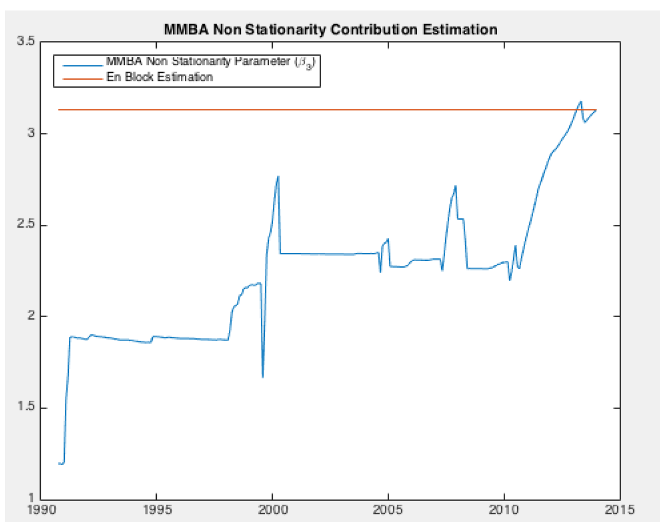

Figure 26: Apple's MMBA BSADFS Parameter ($\widehat{\beta}_3$), from 1990 to 2014

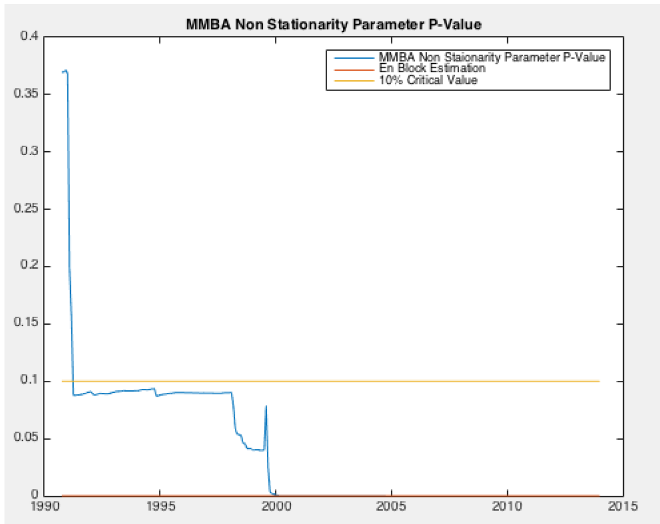

Figure 27: Apple's MMBA BSADFS Parameter T-Statistic P-Value, from 1990 to 2014

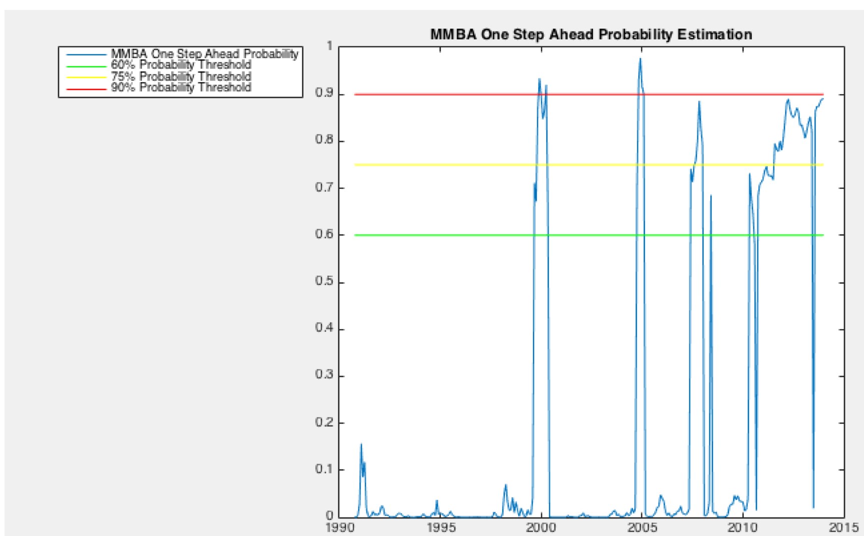

Figure 28: Apple's MMBA One Step Ahead Probability, from 1990 to 2014 $(P_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}])$

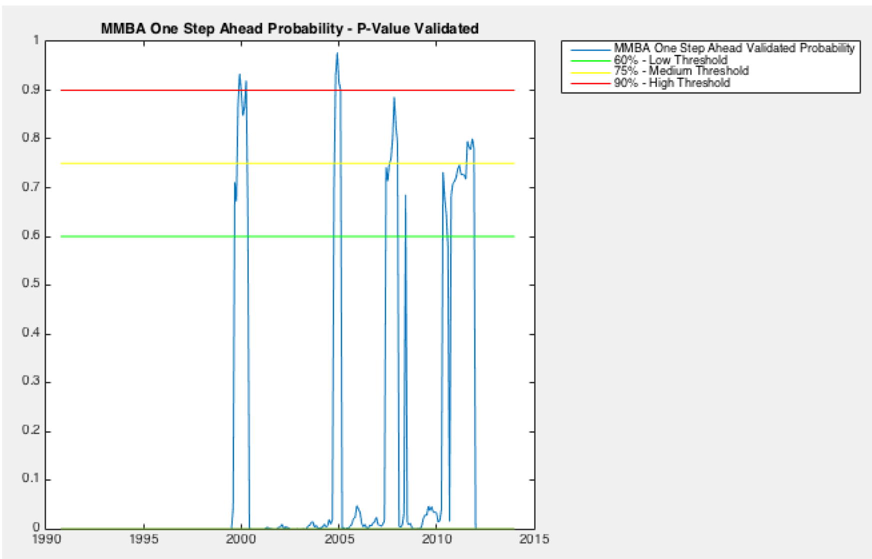

Figure 29: Apple's MMBA One Step Ahead Validated Probability, from 1990 to 2014 ($P^*_{t-1}[\widehat{B_t = 1}|\Omega_{t-1}]$)

7 Further Extensions

There are several forms of extending and improving the application of the newly proposed mechanism, namely the components from which one draws the probability of exuberance: the abnormal return and the non stationarity statistic. At this moment the MMBA can only anticipate exuberance in common stocks. However if one considers a different equilibrium model other than the CAPM, for example the model developed in Fry (2014), then the MMBA becomes completely general. The former framework can calculate for any type of asset in the financial markets, how much of the observed price is due to exuberance. This would allow one to replace the abnormal return in the MMBA's specification and estimate the next period's bubble probability.

Another important aspect is how the subjective rate of return is estimated, that is the choice of the learning rule. For generalization matters the learning rule utilized was basically a recursively updated mean rate of return. However one can consider far more complicated learning rules.

Finally it's also possible to alter the statistic used to capture price acceleration. Nonetheless one should consider both effectiveness and efficiency of the choice, otherwise the MMBA's anticipation power is affected because the identification of possible exuberance starts with unusual price acceleration and the chosen statistic should identify the event with the minimum data consumption. At this moment, the BSADF statistic Phillips et al. (2013) remains the best choice for this job.[11]

[11] Further simulations reveal significant differences in the MMBA predictions when using the BADF Phillips (2012) and its updated version BSADF Phillips et al. (2013)

8 Conclusion

The Moreira-Martins Bubble Anticipator (MMBA) is an econometric mechanism capable of anticipating price exuberance by providing a significant probability at least one step ahead of the bubble peak and whose main novelty, is the incorporation of fundamentals contain in asset pricing theory into the field of bubble testing. Another methodological innovation is the ability to incorporate the social feedback loop Shiller (2000) by the usage of a dynamic econometric specification.

The monte carlo simulations revealed that the fundamentals are quite essential to a better prediction, representing an improvement of 900% when one includes the estimated abnormal return into the econometric specification. The MMBA presents a mean anticipation of 60% for the highest probability threshold (90%) and exhibits an interesting learning dynamic in which the last bubble in the sample is easier to anticipated than all the others. As expected the mechanism presents a higher anticipation power when bubbles are larger in length and scale. For the higher scale bubbles the MMBA was able to predict 100% of the end sample bubbles with the highest probability threshold five periods before the peak.

Finally the application to Apple and NASDAQ Composite monthly data from 1990 to 2014 revealed the MMBA's effectiveness using real data, by providing valid probabilities higher than 85% five months and higher than 90% one month before the "technological bubble" peak, allowing any investor to perform a very profitable surfing strategy.

The new procedure can provide an important contribution for both investors and market regulators due to its Ex Ante nature. This would allow them both to anticipate exuberance one step ahead and act accordingly to their own objectives.

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9 Appendix

9.1 MMBA Monte Carlo Simulation Tables

Table 2: Simulation Variables Description

Variables	Description
	Simulation Overview
P^*/T	Valid MMBA Predictions in Total Sample Size (%)
$\#(B_{1,\dots,T})$	# Of Bubbles Per Simulation
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	# Of Predictions For All Bubbles Per Simulation
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	# Of Predictions For End Sample Bubbles Per Simulation
	All Sample Bubbles Monte Carlo Statistics
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	% Of MMBA One Step Ahead Predictions With Probiblity higher than 60%
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	% Of MMBA One Step Ahead Prediction With Probiblity higher than 75%
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	% Of MMBA One Step Ahead Prediction With Probiblity higher than 90%
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 60%
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 75%
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 90%
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 60%
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 75%
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 90%
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 60%
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 75%
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 90%
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 60%
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 75%
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 90%
	End Sample Bubbles Monte Carlo Statistics
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	% Of MMBA One Step Ahead Prediction With Probiblity higher than 60%
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	% Of MMBA One Step Ahead Prediction With Probiblity higher than 75%
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	% Of MMBA One Step Ahead Prediction With Probiblity higher than 90%
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 60%
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 75%
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	% Of MMBA Two Step Ahead Prediction With Probiblity higher than 90%
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 60%
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 75%
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	% Of MMBA Three Step Ahead Prediction With Probiblity higher than 90%
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 60%
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 75%
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	% Of MMBA Four Step Ahead Prediction With Probiblity higher than 90%
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 60%
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 75%
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	% Of MMBA Five Step Ahead Prediction With Probiblity higher than 90%
	All Sample Bubbles MMBA Probability distribution
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	Probability One Step Ahead Every Bubble Peak
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	Probability Two Steps Ahead Every Bubble Peak
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	Probability Three Steps Ahead Every Bubble Peak
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	Probability Four Steps Ahead Every Bubble Peak
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	Probability Five Steps Ahead Every Bubble Peak
	End Sample Bubbles MMBA Probability distribution
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	Probability One Step Ahead Last Bubble Peak
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	Probability Two Steps Ahead Last Bubble Peak
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	Probability Three Steps Ahead Last Bubble Peak
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	Probability Four Steps Ahead Last Bubble Peak
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	Probability Five Steps Ahead Last Bubble Peak

Table 3: MMBA With Different Sample Sizes, $N = 100$

Variables	Sample Size Monte Carlo Simulation														
Parameters	$T = 200$			$T = 400$			$T = 600$			$T = 800$			$T = 1000$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	8.20	11.34	0.04	30.71	28.85	0.24	57.02	30.84	0.62	74.75	18.77	0.79	84.95	16.42	91.63
$\#(B_{1,...,T})$	1.41	0.53	1.00	2.67	1.05	3.00	3.15	1.19	3.00	4.14	1.39	4.00	4.98	1.79	5.00
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	0.50	0.57	0.00	1.22	1.01	1.00	1.98	0.96	2.00	2.66	1.18	3.00	3.23	1.31	3.00
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.40	0.49	0.00	0.60	0.49	1.00	0.82	0.39	1.00	0.69	0.47	1.00	0.77	0.42	1.00
	All Sample Bubbles Statistics														
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	36.52	43.08	0.00	49.42	38.18	50.00	66.72	29.46	66.67	67.32	25.20	66.67	66.44	23.08	66.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	35.65	42.80	0.00	48.17	39.10	50.00	66.21	29.16	66.67	64.25	27.04	66.67	62.88	25.05	64.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	33.04	41.83	0.00	40.11	37.69	33.33	58.10	30.37	50.00	51.15	29.96	50.00	54.83	26.76	57.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	33.48	41.74	0.00	47.97	37.31	50.00	62.69	31.23	66.00	61.99	27.83	63.33	59.95	24.98	60.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	33.48	41.74	0.00	45.33	38.05	50.00	61.67	31.64	66.00	58.56	28.64	60.00	56.60	26.17	58.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	27.83	39.29	0.00	33.25	36.01	25.00	51.82	31.72	50.00	44.53	30.65	33.33	49.07	27.68	50.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	31.30	41.60	0.00	38.67	36.60	41.67	58.41	32.39	60.00	55.29	27.72	60.00	55.11	25.85	60.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	29.57	40.77	0.00	36.86	36.42	33.33	55.72	32.60	50.00	51.99	29.58	50.00	52.24	25.46	59.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	23.91	38.24	0.00	28.25	33.74	10.00	46.21	33.40	33.00	39.18	30.01	33.33	44.79	27.86	59.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	26.96	39.89	0.00	33.75	36.09	33.33	53.74	33.27	50.00	46.88	30.55	45.00	50.39	26.19	50.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	25.22	38.83	0.00	31.53	34.92	25.00	50.41	33.16	50.00	44.02	31.04	40.00	48.00	26.84	50.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	20.44	36.21	0.00	25.00	33.51	0.00	39.56	36.45	33.00	31.84	30.11	25.00	40.86	28.05	40.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	24.35	39.89	0.00	29.86	36.09	25.00	46.85	33.27	33.00	39.79	30.55	33.33	43.79	26.19	50.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	21.30	38.83	0.00	26.53	34.92	0.00	45.26	33.16	33.00	36.88	31.04	29.17	41.70	26.84	46.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	18.70	36.21	0.00	20.28	33.51	0.00	34.00	36.45	25.00	27.50	30.11	25.00	36.28	28.05	33.00
	End Sample Bubbles Statistics														
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	40.00	49.20	0.00	60.00	49.40	100.00	81.54	39.10	100.00	68.57	46.76	100.00	77.38	42.09	100.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	39.13	49.02	0.00	56.67	49.97	100.00	81.54	39.10	100.00	67.14	47.31	100.00	73.81	44.23	100.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	36.52	48.36	0.00	43.33	49.97	0.00	66.15	47.69	100.00	52.86	50.28	100.00	61.91	48.85	100.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	35.65	48.11	0.00	60.00	49.40	100.00	76.92	42.46	100.00	64.29	48.26	100.00	71.43	45.45	100.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	35.65	48.11	0.00	53.33	50.31	100.00	73.85	44.29	100.00	61.43	49.03	100.00	65.48	47.83	100.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	29.57	45.83	0.00	38.33	49.03	0.00	58.46	49.66	100.00	44.29	50.03	0.00	54.76	50.07	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	33.91	47.55	0.00	45.00	50.17	0.00	69.23	46.51	100.00	58.57	49.62	100.00	61.91	48.85	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	32.17	46.92	0.00	43.33	49.97	0.00	64.62	48.19	100.00	57.14	49.84	100.00	59.52	49.38	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	24.35	43.11	0.00	35.00	48.10	0.00	52.31	50.34	100.00	41.43	49.62	0.00	47.62	50.24	0.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	28.70	45.43	0.00	38.33	49.03	0.00	67.69	47.13	100.00	52.86	58.28	100.00	60.71	49.13	100.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	26.96	44.57	0.00	35.00	48.10	0.00	63.08	48.64	100.00	50.00	58.36	50.00	59.52	49.38	100.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	19.13	39.51	0.00	28.33	45.44	0.00	49.23	50.38	0.00	35.71	48.26	0.00	45.24	50.07	0.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	24.35	45.43	0.00	33.33	49.03	0.00	58.46	47.13	100.00	45.71	50.28	0.00	53.57	49.13	100.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	20.87	44.57	0.00	28.33	48.10	0.00	56.92	48.64	100.00	40.00	50.36	0.00	52.38	49.38	100.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	17.39	39.51	0.00	21.67	45.44	0.00	43.08	50.38	0.00	32.86	48.26	0.00	45.24	50.07	0.00
	All Sample Bubbles MMBA Probability Distribution														
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	95.16	11.56	98.73	78.17	32.89	94.00	76.85	33.97	94.82	72.21	35.88	91.45	73.15	34.85	94.13
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	90.80	19.49	97.60	74.05	34.33	93.00	71.42	36.73	93.01	66.70	37.48	88.62	66.72	37.90	92.36
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	86.46	25.04	96.22	66.22	37.98	89.00	65.66	38.81	89.90	59.01	39.99	83.09	60.97	40.30	88.14
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	81.46	29.72	94.41	58.41	39.76	79.00	59.22	40.77	83.78	58.16	41.60	57.14	55.45	41.39	77.98
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	73.26	34.78	91.41	52.30	41.06	61.00	51.56	42.07	64.70	41.52	41.47	14.01	48.36	42.23	25.28
	End Sample Bubbles MMBA Probability Distribution														
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	95.09	9.83	97.92	79.67	30.44	93.67	82.56	27.74	95.42	68.13	37.48	90.59	77.27	31.14	95.07
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	90.34	19.08	96.61	76.33	31.77	92.89	77.13	32.26	93.96	63.74	39.46	86.83	71.52	34.81	93.04
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	85.99	24.80	95.50	65.88	38.14	89.68	70.57	36.55	92.47	57.85	41.33	83.29	63.97	39.08	88.46
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	80.74	29.54	93.76	57.61	40.34	80.71	67.81	38.16	91.24	51.93	42.55	75.11	61.81	40.09	86.88
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	73.16	34.58	91.24	50.55	41.95	55.52	59.85	41.21	84.53	45.64	42.89	12.88	55.59	42.06	80.13

Table 4: MMBA With Different Initial Sample Sizes, $N = 100$

Variables	MMBA Initial Sample Monte Carlo Simulation								
Parameters	$0\% \times T$			$20\% \times T$			$40\% \times T$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	61.15	19.51	0.65	84.95	16.42	91.63	89.53	13.65	0.95
$\#(B_{1,\ldots,T})$	6.14	2.04	6.00	4.98	1.79	5.00	4.50	1.45	4.50
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	2.98	1.36	3.00	3.23	1.31	3.00	2.93	0.92	3.00
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.73	0.45	1.00	0.77	0.42	1.00	0.50	0.52	0.50
	All Sample Bubbles Statistics								
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	51.01	22.28	50.00	66.44	23.08	66.00	68.55	21.75	69.05
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	49.93	23.18	50.00	62.88	25.05	64.00	63.32	28.70	69.048
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	41.12	25.81	37.50	54.83	26.76	57.00	48.96	31.89	50.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	47.95	22.99	50.00	59.95	24.98	60.00	62.53	22.22	63.33
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	46.27	24.39	50.00	56.60	26.17	58.00	56.11	28.87	58.57
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	35.70	26.01	33.33	49.07	27.68	50.00	36.33	29.40	33.33
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	44.01	23.42	42.86	55.11	25.85	60.00	49.56	28.17	50.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	41.60	24.20	38.75	52.24	25.46	59.00	46.94	31.16	50.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	31.53	24.58	25.00	44.79	27.86	59.00	39.37	28.67	26.79
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	39.32	24.38	35.42	50.39	26.19	50.00	44.56	32.16	50.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	36.43	25.16	33.33	48.00	26.84	50.00	42.77	32.53	45.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	28.34	25.37	20.00	40.86	28.05	40.00	23.76	27.63	18.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	34.80	24.38	30.95	43.79	26.19	50.00	38.18	32.16	41.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	32.29	25.16	25.00	41.70	26.84	46.00	36.99	32.53	41.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	24.31	25.37	16.67	36.28	28.05	33.00	16.97	27.63	15.00
	End Sample Bubbles Statistics								
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	72.73	44.79	100.00	77.38	42.09	100.00	50.00	51.89	50.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	68.18	46.84	100.00	73.81	44.23	100.00	50.00	51.89	50.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	56.82	49.82	100.00	61.91	48.85	100.00	42.86	51.36	0.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	65.91	47.67	100.00	71.43	45.45	100.00	50.00	51.89	50.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	62.50	48.69	100.00	65.48	47.83	100.00	50.00	51.89	50.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	47.73	50.24	0.00	54.76	50.07	100.00	35.71	49.73	0.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	59.09	49.45	100.00	61.91	48.85	100.00	28.57	46.88	0.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	56.82	49.82	100.00	59.52	49.38	100.00	28.57	46.88	0.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	40.91	49.45	0.00	47.62	50.24	0.00	28.57	46.88	0.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	52.27	50.24	100.00	60.71	49.13	100.00	28.57	46.88	0.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	50.00	50.29	50.00	59.52	49.38	100.00	28.57	46.88	0.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	34.09	47.67	0.00	45.24	50.07	0.00	21.43	42.58	0.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	44.32	50.24	0.00	53.57	49.13	100.00	28.57	46.88	0.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	43.18	50.29	0.00	52.38	49.38	100.00	28.57	46.88	0.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	30.68	47.67	0.00	45.24	50.07	0.00	21.43	42.58	0.00
	All Sample Bubbles MMBA Probability Distribution								
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	68.04	39.26	92.00	73.15	34.85	94.13	64.99	38.47	88.28
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	63.58	40.69	88.00	66.72	37.90	92.36	58.77	39.68	82.87
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	57.76	41.95	82.00	60.97	40.30	88.14	49.56	40.83	58.94
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	50.79	42.69	52.00	55.45	41.39	77.98	42.33	41.89	11.43
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	44.81	43.22	16.00	48.36	42.23	25.28	37.24	40.97	9.11
	End Sample Bubbles MMBA Probability Distribution								
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	69.56	36.88	92.48	77.27	31.14	95.07	52.50	43.63	49.05
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	64.87	39.34	88.48	71.52	34.81	93.04	51.10	43.68	46.47
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	57.89	41.56	83.18	63.97	39.08	88.46	35.18	40.89	13.32
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	51.56	42.50	74.82	61.81	40.09	86.88	32.51	41.48	18.17
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	44.53	42.98	17.23	55.59	42.06	80.13	31.40	41.71	8.31

Table 5: MMBA With Different Constant Gains, $N = 100$

Variables	Constant Gains Monte Carlo Simulation								
Parameters	$\gamma = 0.1$			$\gamma = 0.2$			$\gamma = 0.4$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	68.049	30.113	0.73928	84.948	16.424	91.633	85.652	16.231	9.92925
$\#(B_{1,...,T})$	4.0864	1.4849	4	4.9762	1.79	5	5.0988	1.8615	5
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	2.6173	1.2507	3	3.2262	1.3112	3	3.0247	1.3132	3
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.76543	0.42637	1	0.77381	0.42088	1	0.62963	0.48591	1
	All Sample Bubbles Statistics								
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	67.754	28.655	66	66.443	23.08	66	60.982	22.714	60
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	64.976	30.547	66	62.88	25.051	64	56.149	22.594	60
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	59.08	30.731	60	54.83	26.759	57	46.46	25.034	50
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	61.446	29.824	66	59.95	24.979	60	55.117	25.467	50
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	59.491	30.383	60	56.604	26.173	58	52.092	24.36	50
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	53.144	31.927	50	49.069	27.679	50	43.5	27.249	50
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	52.906	31.62	50	55.111	25.851	60	52.241	24.997	50
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	51.671	31.71	50	52.244	25.456	59	48.558	25.35	50
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	45.467	31.539	50	44.788	27.864	59	42.103	26.642	40
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	48.574	31.512	50	50.392	26.194	50	47.405	26.227	42.857
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	46.208	30.992	40	48	26.844	50	43.208	24.702	40
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	40.126	30.962	33.3	40.861	28.048	40	38.843	25.449	33.33
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	44.85	31.512	40	43.788	26.194	50	40.517	26.227	33.33
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	42.381	30.992	33.3	41.704	26.844	46	37.863	24.702	33.33
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	37.184	30.962	33.3	36.282	28.048	33	32.269	25.449	28.571
	End Sample Bubbles Statistics								
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	76.543	42.637	100	77.381	42.088	100	62.963	48.591	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	74.074	44.096	100	73.81	44.231	100	58.025	49.659	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	67.901	46.976	100	61.905	48.854	100	43.21	49.845	0
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	70.37	45.947	100	71.429	45.447	100	55.556	50	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	69.136	46.481	100	65.476	47.83	100	54.321	50.123	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	59.259	49.441	100	54.762	50.072	100	40.741	49.441	0
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	59.259	49.441	100	61.905	48.854	100	50.617	58.308	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	58.025	49.659	100	59.524	49.379	100	48.148	50.277	9
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	51.852	50.277	100	47.619	50.243	0	40.741	49.441	0
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	55.556	50	100	60.714	49.132	100	45.679	50.123	0
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	53.086	50.216	100	59.524	49.379	100	43.21	49.845	0
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	45.679	50.123	0	45.238	50.072	0	38.272	48.908	0
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	50.617	50	100	53.571	49.132	100	39.506	50.123	0
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	49.383	50.216	0	52.381	49.379	100	37.037	49.845	0
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	44.444	50.123	0	45.238	50.072	0	29.63	48.908	0
	All Sample Bubbles MMBA Probability Distribution								
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	74.713	36.329	95.41	73.152	34.853	94.133	67.008	36.464	89.759
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	68.191	39.494	93.5	66.72	37.898	92.362	61.647	38.157	84.894
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	60.475	42.183	90.567	60.966	40.298	88.138	57.156	39.59	72.63
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	54.544	43.316	80.281	55.448	41.394	77.977	52.018	40.475	36.234
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	48.908	43.981	27.174	48.362	42.227	25.28	44.37	40.647	20.203
	End Sample Bubbles MMBA Probability Distribution								
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	77.201	34.712	95.458	77.265	31.138	95.074	66.253	34.849	84.482
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	71.04	38.389	94.151	71.519	34.805	93.044	60.475	37.452	80.879
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	61.624	42.165	91.449	63.967	39.079	88.462	55.979	39.21	62.631
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	57.005	43.679	88.569	61.812	40.093	86.876	51.511	40.09	45.772
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	52.114	44.953	84.679	55.585	42.064	80.131	44.205	40.904	21.7

Table 6: MMBA Without Asset Pricing Equilibrium Model, $N = 100$

Variables	Asset Pricing Contribution Monte Carlo Simulation					
Parameters	No Equilibrium Model ($\beta_2 = 0$)			Full MMBA		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	97.76	3.1228	0.98503	84.948	16.424	91.633
$\#(B_{1,...,T})$	5.2763	1.7405	5	4.9762	1.79	5
$\#(P^*_{t-1}[B_t]\vert\Omega_{t-1}] \geq 60)$	3.3026	1.1433	3	3.2262	1.3112	3
$\#(P^*_{T-1}[B_T\vert\Omega_{T-1}] \geq 60)$	0.72368	0.45015	1	0.77381	0.42088	1
	All Sample Bubbles Statistics					
$P^*_{t-1}[B_t = 1\vert\Omega_{t-1}] \geq 60\%$	66.825	21.957	66.667	66.443	23.08	66
$P^*_{t-1}[B_t = 1\vert\Omega_{t-1}] \geq 75\%$	64.695	23.888	66.667	62.88	25.051	64
$P^*_{t-1}[B_t = 1\vert\Omega_{t-1}] \geq 90\%$	17.008	27.235	0	54.83	26.759	57
$P^*_{t-2}[B_t = 1\vert\Omega_{t-2}] \geq 60\%$	62.409	22.428	61.25	59.95	24.979	60
$P^*_{t-2}[B_t = 1\vert\Omega_{t-2}] \geq 75\%$	59.687	24.079	60	56.604	26.173	58
$P^*_{t-2}[B_t = 1\vert\Omega_{t-2}] \geq 90\%$	14.346	24.961	0	49.069	27.679	50
$P^*_{t-3}[B_t = 1\vert\Omega_{t-3}] \geq 60\%$	57.251	25.093	52.778	55.111	25.851	60
$P^*_{t-3}[B_t = 1\vert\Omega_{t-3}] \geq 75\%$	54.794	27.047	50	52.244	25.456	59
$P^*_{t-3}[B_t = 1\vert\Omega_{t-3}] \geq 90\%$	13.235	24.825	0	44.788	27.864	59
$P^*_{t-4}[B_t = 1\vert\Omega_{t-4}] \geq 60\%$	50.634	24.724	50	50.392	26.194	50
$P^*_{t-4}[B_t = 1\vert\Omega_{t-4}] \geq 75\%$	46.971	26.627	50	48	26.844	50
$P^*_{t-4}[B_t = 1\vert\Omega_{t-4}] \geq 90\%$	11.272	23.058	0	40.861	28.048	40
$P^*_{t-5}[B_t = 1\vert\Omega_{t-5}] \geq 60\%$	43.755	24.724	40	43.788	26.194	50
$P^*_{t-5}[B_t = 1\vert\Omega_{t-5}] \geq 75\%$	41.146	26.627	40	41.704	26.844	46
$P^*_{t-5}[B_t = 1\vert\Omega_{t-5}] \geq 90\%$	9.9781	23.058	0	36.282	28.048	33
	End Sample Bubbles Statistics					
$P^*_{T-1}[B_T = 1\vert\Omega_{T-1}] \geq 60\%$	72.368	45.015	100	77.381	42.088	100
$P^*_{T-1}[B_T = 1\vert\Omega_{T-1}] \geq 75\%$	71.053	45.653	100	73.81	44.231	100
$P^*_{T-1}[B_T = 1\vert\Omega_{T-1}] \geq 90\%$	10.526	30.893	0	61.905	48.854	100
$P^*_{T-2}[B_T = 1\vert\Omega_{T-2}] \geq 60\%$	64.474	48.177	100	71.429	45.447	100
$P^*_{T-2}[B_T = 1\vert\Omega_{T-2}] \geq 75\%$	63.158	48.558	100	65.476	47.83	100
$P^*_{T-2}[B_T = 1\vert\Omega_{T-2}] \geq 90\%$	10.526	30.893	0	54.762	50.072	100
$P^*_{T-3}[B_T = 1\vert\Omega_{T-3}] \geq 60\%$	59.211	49.471	100	61.905	48.854	100
$P^*_{T-3}[B_T = 1\vert\Omega_{T-3}] \geq 75\%$	57.895	49.701	100	59.524	49.379	100
$P^*_{T-3}[B_T = 1\vert\Omega_{T-3}] \geq 90\%$	7.8947	27.145	0	47.619	50.243	0
$P^*_{T-4}[B_T = 1\vert\Omega_{T-4}] \geq 60\%$	50	50.332	50	60.714	49.132	100
$P^*_{T-4}[B_T = 1\vert\Omega_{T-4}] \geq 75\%$	48.684	50.315	0	59.524	49.379	100
$P^*_{T-4}[B_T = 1\vert\Omega_{T-4}] \geq 90\%$	5.2632	22.478	0	45.238	50.072	0
$P^*_{T-5}[B_T = 1\vert\Omega_{T-5}] \geq 60\%$	42.105	50.332	0	53.571	49.132	100
$P^*_{T-5}[B_T = 1\vert\Omega_{T-5}] \geq 75\%$	40.789	50.315	0	52.381	49.379	100
$P^*_{T-5}[B_T = 1\vert\Omega_{T-5}] \geq 90\%$	5.2632	22.478	0	45.238	50.072	0
	All Sample Bubbles MMBA Probability Distribution					
$P^*_{t-1}[B_t\vert\Omega_{t-1}]$	56.22	40.952	83.333	73.152	34.853	94.133
$P^*_{t-2}[B_t\vert\Omega_{t-2}]$	51.973	41.849	80.952	66.72	37.898	92.362
$P^*_{t-3}[B_t\vert\Omega_{t-3}]$	46.49	42.697	76.923	60.966	40.298	88.138
$P^*_{t-4}[B_t\vert\Omega_{t-4}]$	40.77	42.551	1.2626	55.448	41.394	77.977
$P^*_{t-5}[B_t\vert\Omega_{t-5}]$	34.675	41.614	1.0955	48.362	42.227	25.28
	End Sample Bubbles MMBA Probability Distribution					
$P^*_{T-1}[B_T\vert\Omega_{T-1}]$	61.988	38.207	83.827	77.265	31.138	95.074
$P^*_{T-2}[B_T\vert\Omega_{T-2}]$	55.272	40.807	82.64	71.519	34.805	93.044
$P^*_{T-3}[B_T\vert\Omega_{T-3}]$	50.825	41.87	80.278	63.967	39.079	88.462
$P^*_{T-4}[B_T\vert\Omega_{T-4}]$	43.16	42.704	37.047	61.812	40.093	86.876
$P^*_{T-5}[B_T\vert\Omega_{T-5}]$	36.436	42.128	1.1143	55.585	42.064	80.131

Table 7: MMBA Without P-Value Validation, $N = 100$

Variables	P-Value Validation Monte Carlo Simulation					
Parameters	Unvalidated MMBA			Validated MMBA		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	84.48	15.334	0.87755	84.948	16.424	91.633
$\#(B_{1,\dots,T})$	5.3218	1.8708	5	4.9762	1.79	5
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	3.3793	1.2223	3	3.2262	1.3112	3
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.64368	0.48169	1	0.77381	0.42088	1
	All Sample Bubbles Statistics					
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	66.318	20.901	66.667	66.443	23.08	66
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	60.87	21.08	62.5	62.88	25.051	64
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	50.502	22.039	55.556	54.83	26.759	57
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	61.003	23.785	60	59.95	24.979	60
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	55.746	22.194	57.143	56.604	26.173	58
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	45.744	22.618	50	49.069	27.679	50
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	55.976	25.123	50	55.111	25.851	60
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	52.036	24.835	50	52.244	25.456	59
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	41.428	24.08	42.857	44.788	27.864	59
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	48.983	28.713	42.857	50.392	26.194	50
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	44.822	27.172	42.857	48	26.844	50
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	37.935	25.688	40	40.861	28.048	40
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	44.43	28.713	40	43.788	26.194	50
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	40.866	27.172	40	41.704	26.844	46
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	34.252	25.688	33.333	36.282	28.048	33
	End Sample Bubbles Statistics					
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	64.368	48.169	100	77.381	42.088	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	59.77	49.32	100	73.81	44.231	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	47.126	50.207	0	61.905	48.854	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	58.621	49.537	100	71.429	45.447	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	55.172	50.02	100	65.476	47.83	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	42.529	49.725	0	54.762	50.072	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	55.172	50.02	100	61.905	48.854	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	50.575	50.287	100	59.524	49.379	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	37.931	48.803	0	47.619	50.243	0
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	49.425	50.287	0	60.714	49.132	100
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	42.529	49.725	0	59.524	49.379	100
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	36.782	48.501	0	45.238	50.072	0
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	42.529	50.287	0	53.571	49.132	100
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	40.23	49.725	0	52.381	49.379	100
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	33.333	48.501	0	45.238	50.072	0
	All Sample Bubbles MMBA Probability Distribution					
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	65.698	37.505	90.259	73.152	34.853	94.133
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	60.044	38.945	83.713	66.72	37.898	92.362
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	54.602	40.263	66.966	60.966	40.298	88.138
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	46.989	41.127	25.164	55.448	41.394	77.977
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	41.391	41.276	17.556	48.362	42.227	25.28
	End Sample Bubbles MMBA Probability Distribution					
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	65.036	38.515	89.161	77.265	31.138	95.074
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	60.438	39.109	82.718	71.519	34.805	93.044
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	56.505	40.217	75.483	63.967	39.079	88.462
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	50.114	41.468	30.732	61.812	40.093	86.876
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	44.629	41.811	21.966	55.585	42.064	80.131

Table 8: MMBA With Different Average Length Thresholds (α), $N = 100$

Variables	Bubble Trigger Monte Carlo Simulation								
Parameters	$\alpha = 0.1$			$\alpha = 1$			$\alpha = 10$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	80.28	20.33	0.87	84.95	16.42	91.63	65.93	31.68	0.76
$\#(B_{1,\ldots,T})$	5.47	2.31	6.00	4.98	1.79	5.00	5.49	2.07	6.00
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	3.02	1.46	3.00	3.23	1.31	3.00	3.28	2.39	3.00
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.54	0.50	1.00	0.77	0.42	1.00	0.75	0.44	1.00
	All Sample Bubbles Statistics								
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	60.77	28.20	60.00	66.44	23.08	66.00	55.75	34.88	60.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	58.14	28.42	56.35	62.88	25.05	64.00	55.75	34.88	60.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	46.83	30.93	42.85	54.83	26.76	57.00	34.73	25.54	35.42
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	54.82	29.77	50.00	59.95	24.98	60.00	56.74	35.42	64.58
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	51.71	30.60	50.00	56.60	26.17	58.00	56.74	35.42	64.58
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	40.88	30.74	33.00	49.07	27.68	50.00	40.32	27.79	46.43
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	50.77	30.46	44.00	55.11	25.85	60.00	57.73	35.03	66.66
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	47.42	31.18	40.00	52.24	25.46	59.00	57.73	35.03	66.66
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	38.41	30.67	33.00	44.79	27.86	59.00	42.59	29.04	50.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	45.47	31.57	34.00	50.39	26.19	50.00	59.09	35.92	66.66
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	43.73	31.97	33.00	48.00	26.84	50.00	58.59	35.70	66.66
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	34.68	30.55	25.00	40.86	28.05	40.00	45.72	30.71	50.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	40.88	31.57	33.00	43.79	26.19	50.00	59.94	35.92	71.42
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	38.87	31.97	33.00	41.70	26.84	46.00	59.94	35.70	71.42
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	31.80	30.55	25.00	36.28	28.05	33.00	48.13	30.71	50.00
	End Sample Bubbles Statistics								
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	54.00	50.09	100.00	77.38	42.09	100.00	75.00	43.52	100.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	50.00	50.25	50.00	73.81	44.23	100.00	75.00	43.52	100.00
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	31.00	46.48	0.00	61.91	48.85	100.00	44.00	49.89	0.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	47.00	50.16	0.00	71.43	45.45	100.00	76.00	42.92	100.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	42.00	49.60	0.00	65.48	47.83	100.00	76.00	42.92	100.00
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	24.00	42.92	0.00	54.76	50.07	100.00	57.00	49.76	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	41.00	49.43	0.00	61.91	48.85	100.00	76.00	42.92	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	37.00	48.52	0.00	59.52	49.38	100.00	76.00	42.92	100.00
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	24.00	42.92	0.00	47.62	50.24	0.00	57.00	49.76	100.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	35.00	47.94	0.00	60.71	49.13	100.00	76.00	42.92	100.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	34.00	47.61	0.00	59.52	49.38	100.00	76.00	42.92	100.00
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	22.00	41.63	0.00	45.24	50.07	0.00	61.00	49.02	100.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	31.00	47.94	0.00	53.57	49.13	100.00	77.00	42.92	100.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	27.00	47.61	0.00	52.38	49.38	100.00	77.00	42.92	100.00
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	20.00	41.63	0.00	45.24	50.07	0.00	65.00	49.02	100.00
	All Sample Bubbles MMBA Probability Distribution								
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	60.41	39.85	83.68	73.15	34.85	94.13	92.46	5.33	93.05
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	53.54	41.27	71.82	66.72	37.90	92.36	92.96	5.34	93.77
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	47.80	42.08	23.42	60.97	40.30	88.14	93.44	5.35	94.60
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	42.30	42.10	13.98	55.45	41.39	77.98	93.91	5.33	95.21
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	37.37	41.34	9.14	48.36	42.23	25.28	94.12	6.62	95.58
	End Sample Bubbles MMBA Probability Distribution								
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	63.13	37.99	84.15	77.27	31.14	95.07	91.67	3.46	92.08
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	55.43	40.25	76.95	71.52	34.81	93.04	92.20	3.46	92.56
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	48.49	42.28	24.33	63.97	39.08	88.46	92.72	3.42	93.21
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	43.41	42.26	15.82	61.81	40.09	86.88	93.31	3.42	94.23
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	37.74	41.74	6.94	55.59	42.06	80.13	93.82	3.39	95.04

Table 9: MMBA With Different Frequencies of Bubble Eruptions (δ), $N = 100$

Variables	Bubble Restarting Value Monte Carlo Simulation								
Parameters	$\delta = 0.05$			$\delta = 0.5$			$\delta = 5$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	61.15	19.51	0.65	84.95	16.42	91.63	80.17	17.61	0.87
$\#(B_{1,...,T})$	6.14	2.04	6.00	4.98	1.79	5.00	8.98	3.31	9.00
$\#(P^*_{t-1}[B_t] \vert \Omega_{t-1}] \geq 60)$	2.98	1.36	3.00	3.23	1.31	3.00	3.45	1.51	4.00
$\#(P^*_{T-1}[B_T \vert \Omega_{T-1}] \geq 60)$	0.73	0.45	1.00	0.77	0.42	1.00	0.45	0.50	0.00
All Sample Bubbles Statistics									
$P^*_{t-1}[B_t = 1 \vert \Omega_{t-1}] \geq 60\%$	51.01	22.28	50.00	66.44	23.08	66.00	45.81	27.69	40.00
$P^*_{t-1}[B_t = 1 \vert \Omega_{t-1}] \geq 75\%$	49.93	23.18	50.00	62.88	25.05	64.00	43.94	28.43	33.33
$P^*_{t-1}[B_t = 1 \vert \Omega_{t-1}] \geq 90\%$	41.12	25.81	37.50	54.83	26.76	57.00	32.70	28.26	25.00
$P^*_{t-2}[B_t = 1 \vert \Omega_{t-2}] \geq 60\%$	47.95	22.99	50.00	59.95	24.98	60.00	39.91	27.47	33.33
$P^*_{t-2}[B_t = 1 \vert \Omega_{t-2}] \geq 75\%$	46.27	24.39	50.00	56.60	26.17	58.00	37.75	27.80	33.33
$P^*_{t-2}[B_t = 1 \vert \Omega_{t-2}] \geq 90\%$	35.70	26.01	33.33	49.07	27.68	50.00	29.06	26.68	22.22
$P^*_{t-3}[B_t = 1 \vert \Omega_{t-3}] \geq 60\%$	44.01	23.42	42.86	55.11	25.85	60.00	36.02	27.99	30.00
$P^*_{t-3}[B_t = 1 \vert \Omega_{t-3}] \geq 75\%$	41.60	24.20	38.75	52.24	25.46	59.00	33.91	28.23	25.00
$P^*_{t-3}[B_t = 1 \vert \Omega_{t-3}] \geq 90\%$	31.53	24.58	25.00	44.79	27.86	59.00	26.72	26.35	20.00
$P^*_{t-4}[B_t = 1 \vert \Omega_{t-4}] \geq 60\%$	39.32	24.38	35.42	50.39	26.19	50.00	32.41	27.75	25.00
$P^*_{t-4}[B_t = 1 \vert \Omega_{t-4}] \geq 75\%$	36.43	25.16	33.33	48.00	26.84	50.00	30.36	27.46	20.00
$P^*_{t-4}[B_t = 1 \vert \Omega_{t-4}] \geq 90\%$	28.34	25.37	20.00	40.86	28.05	40.00	24.55	26.56	14.30
$P^*_{t-5}[B_t = 1 \vert \Omega_{t-5}] \geq 60\%$	34.80	24.38	30.95	43.79	26.19	50.00	27.73	27.75	20.00
$P^*_{t-5}[B_t = 1 \vert \Omega_{t-5}] \geq 75\%$	32.29	25.16	25.00	41.70	26.84	46.00	26.49	27.46	20.00
$P^*_{t-5}[B_t = 1 \vert \Omega_{t-5}] \geq 90\%$	24.31	25.37	16.67	36.28	28.05	33.00	20.76	26.56	11.11
End Sample Bubbles Statistics									
$P^*_{T-1}[B_T = 1 \vert \Omega_{T-1}] \geq 60\%$	72.73	44.79	100.00	77.38	42.09	100.00	45.16	50.04	0.00
$P^*_{T-1}[B_T = 1 \vert \Omega_{T-1}] \geq 75\%$	68.18	46.84	100.00	73.81	44.23	100.00	43.01	49.78	0.00
$P^*_{T-1}[B_T = 1 \vert \Omega_{T-1}] \geq 90\%$	56.82	49.82	100.00	61.91	48.85	100.00	27.96	45.12	0.00
$P^*_{T-2}[B_T = 1 \vert \Omega_{T-2}] \geq 60\%$	65.91	47.67	100.00	71.43	45.45	100.00	36.56	48.42	0.00
$P^*_{T-2}[B_T = 1 \vert \Omega_{T-2}] \geq 75\%$	62.50	48.69	100.00	65.48	47.83	100.00	33.33	47.40	0.00
$P^*_{T-2}[B_T = 1 \vert \Omega_{T-2}] \geq 90\%$	47.73	50.24	0.00	54.76	50.07	100.00	24.73	43.38	0.00
$P^*_{T-3}[B_T = 1 \vert \Omega_{T-3}] \geq 60\%$	59.09	49.45	100.00	61.91	48.85	100.00	33.33	47.40	0.00
$P^*_{T-3}[B_T = 1 \vert \Omega_{T-3}] \geq 75\%$	56.82	49.82	100.00	59.52	49.38	100.00	32.26	47.00	0.00
$P^*_{T-3}[B_T = 1 \vert \Omega_{T-3}] \geq 90\%$	40.91	49.45	0.00	47.62	50.24	0.00	19.36	39.72	0.00
$P^*_{T-4}[B_T = 1 \vert \Omega_{T-4}] \geq 60\%$	52.27	50.24	100.00	60.71	49.13	100.00	30.11	46.12	0.00
$P^*_{T-4}[B_T = 1 \vert \Omega_{T-4}] \geq 75\%$	50.00	50.29	50.00	59.52	49.38	100.00	27.96	45.12	0.00
$P^*_{T-4}[B_T = 1 \vert \Omega_{T-4}] \geq 90\%$	34.09	47.67	0.00	45.24	50.07	0.00	17.20	37.95	0.00
$P^*_{T-5}[B_T = 1 \vert \Omega_{T-5}] \geq 60\%$	44.32	50.24	0.00	53.57	49.13	100.00	26.88	46.12	0.00
$P^*_{T-5}[B_T = 1 \vert \Omega_{T-5}] \geq 75\%$	43.18	50.29	0.00	52.38	49.38	100.00	23.66	45.12	0.00
$P^*_{T-5}[B_T = 1 \vert \Omega_{T-5}] \geq 90\%$	30.68	47.67	0.00	45.24	50.07	0.00	15.05	37.95	0.00
All Sample Bubbles MMBA Probability Distribution									
$P^*_{t-1}[B_t \vert \Omega_{t-1}]$	68.04	39.26	92.00	73.15	34.85	94.13	43.98	40.49	18.09
$P^*_{t-2}[B_t \vert \Omega_{t-2}]$	63.58	40.69	88.00	66.72	37.90	92.36	37.78	39.97	11.95
$P^*_{t-3}[B_t \vert \Omega_{t-3}]$	57.76	41.95	82.00	60.97	40.30	88.14	32.90	39.21	8.00
$P^*_{t-4}[B_t \vert \Omega_{t-4}]$	50.79	42.69	52.00	55.45	41.39	77.98	28.58	38.03	6.08
$P^*_{t-5}[B_t \vert \Omega_{t-5}]$	44.81	43.22	16.00	48.36	42.23	25.28	24.51	36.38	4.17
End Sample Bubbles MMBA Probability Distribution									
$P^*_{T-1}[B_T \vert \Omega_{T-1}]$	69.56	36.88	92.48	77.27	31.14	95.07	46.60	41.23	18.85
$P^*_{T-2}[B_T \vert \Omega_{T-2}]$	64.87	39.34	88.48	71.52	34.81	93.04	38.41	40.97	10.15
$P^*_{T-3}[B_T \vert \Omega_{T-3}]$	57.89	41.56	83.18	63.97	39.08	88.46	35.55	41.03	7.77
$P^*_{T-4}[B_T \vert \Omega_{T-4}]$	51.56	42.50	74.82	61.81	40.09	86.88	31.48	40.10	6.63
$P^*_{T-5}[B_T \vert \Omega_{T-5}]$	44.53	42.98	17.23	55.59	42.06	80.13	27.94	38.92	4.00

Table 10: MMBA With Different Bubble Scales (π), $N = 100$

Variables	Bubble Scale Monte Carlo Simulation								
Parameters	$\pi = 0.5$			$\pi = 0.85$			$\pi = 0.99$		
Statistics	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$	$\bar{X}$	ΣX	$Med(X)$
P^*/T	16.75	20.93	0.09	84.95	16.42	91.63	80.40	23.70	0.85
$\#(B_{1,...,T})$	4.27	1.62	4.00	4.98	1.79	5.00	1.63	0.74	1.50
$\#(P^*_{t-1}[B_t]\|\Omega_{t-1}] \geq 60)$	0.16	0.37	8.00	3.23	1.31	3.00	1.25	0.46	1.00
$\#(P^*_{T-1}[B_T\|\Omega_{T-1}] \geq 60)$	0.08	0.27	0.00	0.77	0.42	1.00	1.00	0.00	1.00
	All Sample Bubbles Statistics								
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 60\%$	4.595	11.164	0.000	66.44	23.08	66.00	83.33	23.57	100.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 75\%$	3.041	9.906	0.000	62.88	25.05	64.00	83.33	23.57	100.00
$P^*_{t-1}[B_t = 1\|\Omega_{t-1}] \geq 90\%$	1.464	6.375	0.000	54.83	26.76	57.00	83.33	23.57	100.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 60\%$	2.140	8.510	0.000	59.95	24.98	60.00	83.33	23.57	100.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 75\%$	1.239	6.119	0.000	56.60	26.17	58.00	83.33	23.57	100.00
$P^*_{t-2}[B_t = 1\|\Omega_{t-2}] \geq 90\%$	0.450	3.875	0.000	49.07	27.68	50.00	83.33	23.57	100.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 60\%$	0.450	3.875	0.000	55.11	25.85	60.00	83.33	23.57	100.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 75\%$	0.450	3.875	0.000	52.24	25.46	59.00	83.33	23.57	100.00
$P^*_{t-3}[B_t = 1\|\Omega_{t-3}] \geq 90\%$	0.000	0.000	0.000	44.79	27.86	59.00	83.33	23.57	100.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 60\%$	0.450	3.875	0.000	50.39	26.19	50.00	83.33	23.57	100.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 75\%$	0.000	0.000	0.000	48.00	26.84	50.00	83.33	23.57	100.00
$P^*_{t-4}[B_t = 1\|\Omega_{t-4}] \geq 90\%$	0.000	0.000	0.000	40.86	28.05	40.00	83.33	23.57	100.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 60\%$	0.450	3.875	0.000	43.79	26.19	50.00	83.33	23.57	100.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 75\%$	0.000	0.000	0.000	41.70	26.84	46.00	83.33	23.57	100.00
$P^*_{t-5}[B_t = 1\|\Omega_{t-5}] \geq 90\%$	0.000	0.000	0.000	36.28	28.05	33.00	83.33	23.57	100.00
	End Sample Bubbles Statistics								
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 60\%$	8.108	27.482	0.000	77.38	42.09	100.00	100	0	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 75\%$	6.757	25.272	0.000	73.81	44.23	100.00	100	0	100
$P^*_{T-1}[B_T = 1\|\Omega_{T-1}] \geq 90\%$	4.054	19.857	0.000	61.91	48.85	100.00	100	0	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 60\%$	5.405	22.767	0.000	71.43	45.45	100.00	100	0	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 75\%$	4.054	19.857	0.000	65.48	47.83	100.00	100	0	100
$P^*_{T-2}[B_T = 1\|\Omega_{T-2}] \geq 90\%$	1.351	11.625	0.000	54.76	50.07	100.00	100	0	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 60\%$	1.351	11.625	0.000	61.91	48.85	100.00	100	0	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 75\%$	1.351	11.625	0.000	59.52	49.38	100.00	100	0	100
$P^*_{T-3}[B_T = 1\|\Omega_{T-3}] \geq 90\%$	0.000	0.000	0.000	47.62	50.24	0.00	100	0	100
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 60\%$	1.351	11.625	0.000	60.71	49.13	100.00	100	0	100
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 75\%$	0.000	0.000	0.000	59.52	49.38	100.00	100	0	100
$P^*_{T-4}[B_T = 1\|\Omega_{T-4}] \geq 90\%$	0.000	0.000	0.000	45.24	50.07	0.00	100	0	100
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 60\%$	1.351	11.625	0.000	53.57	49.13	100.00	100	0	100
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 75\%$	0.000	0.000	0.000	52.38	49.38	100.00	100	0	100
$P^*_{T-5}[B_T = 1\|\Omega_{T-5}] \geq 90\%$	0.000	0.000	0.000	45.24	50.07	0.00	100	0	100
	All Sample Bubbles MMBA Probability Distribution								
$P^*_{t-1}[B_t\|\Omega_{t-1}]$	27.60	34.15	6.08	73.15	34.85	94.13	98.93	0.97	99.25
$P^*_{t-2}[B_t\|\Omega_{t-2}]$	14.50	28.40	1.34	66.72	37.90	92.36	99.01	0.74	99.19
$P^*_{t-3}[B_t\|\Omega_{t-3}]$	7.66	21.19	0.77	60.97	40.30	88.14	98.99	0.85	99.19
$P^*_{t-4}[B_t\|\Omega_{t-4}]$	3.61	13.47	0.71	55.45	41.39	77.98	99.02	0.76	99.12
$P^*_{t-5}[B_t\|\Omega_{t-5}]$	2.30	10.06	0.67	48.36	42.23	25.28	99.14	0.60	99.38
	End Sample Bubbles MMBA Probability Distribution								
$P^*_{T-1}[B_T\|\Omega_{T-1}]$	29.65	37.44	4.18	77.27	31.14	95.07	98.89	0.61	99.01
$P^*_{T-2}[B_T\|\Omega_{T-2}]$	19.45	32.23	1.38	71.52	34.81	93.04	98.99	0.76	99.07
$P^*_{T-3}[B_T\|\Omega_{T-3}]$	10.68	23.08	0.88	63.97	39.08	88.46	99.09	0.69	99.32
$P^*_{T-4}[B_T\|\Omega_{T-4}]$	4.94	15.49	0.82	61.81	40.09	86.88	99.12	0.56	99.26
$P^*_{T-5}[B_T\|\Omega_{T-5}]$	2.58	8.55	0.75	55.59	42.06	80.13	99.21	0.59	99.37

9.2 Mathematical Demonstrations

9.2.1 Obtaining the General Asset Pricing Equation

The former problem can be solved using dynamic programming, the Bellman equation is presented next:

$$V(x_t) \equiv \max_{\{c_t\}}\{u(c_t) + \beta E_t V(x_{t+1})\} \tag{24}$$

Next, we obtain the first order condition:

$$\frac{\partial V(x_t)}{\partial c_t} = 0 \Longleftrightarrow u'(c_t) + \beta \frac{\partial x_{t+1}}{\partial c_t} E_t[V'(x_{t+1})] = 0 \tag{25}$$

We also need to evaluate the evolution of the state variable (x_t):

$$V'(x_t) \equiv \frac{\partial V(x_t)}{\partial x_t} = \beta \frac{\partial x_{t+1}}{\partial x_t} E_t[V'(x_{t+1})] \tag{26}$$

Our objective is to arrive at the Euler equation, for that we rearrange 25

$$E_t[V'(x_{t+1})] = -\frac{u'(c_t)}{\beta \frac{\partial x_{t+1}}{\partial c_t}} \tag{27}$$

And now we plug on 26:

$$V'(x_t) \equiv \frac{\partial V(x_t)}{\partial x_t} = \beta \frac{\partial x_{t+1}}{\partial x_t} \times (-\frac{u'(c_t)}{\beta \frac{\partial x_{t+1}}{\partial c_t}}) = u'(c_t) \times \frac{\frac{\partial x_{t+1}}{\partial x_t}}{\frac{\partial x_{t+1}}{\partial c_t}} \tag{28}$$

Its possible to advance the equation 28 one period forward, and we obtain:

$$V'(x_{t+1}) \equiv \frac{\partial V(x_{t+1})}{\partial x_{t+1}} = u'(c_{t+1}) \times \frac{\frac{\partial x_{t+2}}{\partial x_{t+1}}}{\frac{\partial x_{t+2}}{\partial c_{t+1}}} \tag{29}$$

Plugin 28 and 29 in 26, we obtain the following:

$$u'(c_t) \times \frac{\frac{\partial x_{t+1}}{\partial x_t}}{\frac{\partial x_{t+1}}{\partial c_t}} = \beta \frac{\partial x_{t+1}}{\partial x_t} \times E_t[u'(c_{t+1}) \times \frac{\frac{\partial x_{t+2}}{\partial x_{t+1}}}{\frac{\partial x_{t+2}}{\partial c_{t+1}}}] \tag{30}$$

Know, we take derivatives of proper functional form, but first we rearrange the restriction of the problem, assuming we are in period t:

$$c_t - y_t = (p_t + d_t)x_t - p_t x_{t+1} \Longleftrightarrow x_{t+1} = \frac{-c_t}{p_t} + \frac{y_t}{p_t} + \frac{(p_t + d_t)}{p_t} x_t \tag{31}$$

Now we take the required derivatives:

$$\frac{\partial x_{t+1}}{\partial c_t} = -\frac{1}{p_t} \tag{32}$$

$$\frac{\partial x_{t+1}}{\partial x_t} = -\frac{p_t + d_t}{p_t} \tag{33}$$

Plugging back 32 and 33 in 30, and after rearranging we will obtain the Euler equation:

$$u'(c_t) \times \frac{-\frac{p_t+d_t}{p_t}}{-\frac{1}{p_t}} = \beta \times (-\frac{p_t + d_t}{p_t}) E_t[u'(c_{t+1}) \times \frac{-\frac{p_{t+1}+d_{t+1}}{p_{t+1}}}{-\frac{1}{p_{t+1}}}] \tag{34}$$

And we will obtain the same Euler equation as in Gurkaynak (2005):

$$u'(c_t)p_t = E_t[\beta u'(c_{t+1}) \times x_{t+1}] \Longleftrightarrow p_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}] \tag{35}$$

And finally the general asset pricing equation given in Cochrane (2005):

$$p_t = E_t[m_{t+1} x_{t+1}] \tag{36}$$

9.2.2 From The General Pricing Equation to Lucas (1978)

Assuming the linearity of the investor's utility function, implying: $u'(c_t) = u'(c_{t+1}) = c \in \Re$, *we obtain the asset pricing equation presented in Lucas (1978).*

$$P_t = E_t[m_{t+1}x_{t+1}] \Longleftrightarrow P_t = E_t[\underbrace{(\frac{1}{1+r})}_{\beta} \times \frac{u'(c_{t+1})}{u'(c_t)} \times (P_{t+1} + D_{t+1})]$$

$$\Longleftrightarrow P_t = (\tfrac{1}{1+r})E_t[(\tfrac{c}{c}) \times P_{t+1} + D_{t+1}] \Longleftrightarrow P_t = (\tfrac{1}{1+r})E_t[P_{t+1} + D_{t+1}]$$

9.2.3 The Reduced Form Of The Price Process

Assuming the dividends can be represented by a stochastic process such as: $D_t = \mu + D_{t-1} + \epsilon_t$, $\epsilon_t \sim N(o, \sigma^2)$, the price process will be given by: $P_t = (\frac{1+r}{r^2})\mu + (\frac{1}{r})D_t$

9.2.4 Bubbles Are Consistent With R.E

Equation (7) constitutes a valid solution to the difference equation given by (3), if the stochastic process driving the bubble component (B_t) *respects the following specification:*

$$B_{t+1} = (1+r)B_t + \epsilon \Longrightarrow E_t[B_{t+1}] = (1+r)B_t \tag{37}$$

$$P_t = (\tfrac{1}{1+r})E_t[P_{t+1} + D_{t+1}] \Longleftrightarrow$$

$$\begin{aligned}
&F_t + B_t = (\tfrac{1}{1+r})(E_t[F_{t+1}] + E_t[B_{t+1}] + E_t[D_{t+1}]) \Longleftrightarrow \\
&F_t + B_t = (\tfrac{1}{1+r})(E_t[F_{t+1}] + (1+r)B_t + E_t[D_{t+1}]) \Longleftrightarrow \\
&F_t + B_t - B_t = (\tfrac{1}{1+r})(E_t[F_{t+1}] + E_t[D_{t+1}]) \Longleftrightarrow \\
&F_t = (\tfrac{1}{1+r})(E_t[F_{t+1}] + E_t[D_{t+1}]) \Longleftrightarrow \\
&F_t = \textstyle\sum_{i=1}^{+\infty}[(\tfrac{1}{1+r})^i \times E_t[D_{t+i}]] = P_t
\end{aligned}$$

9.2.5 Solving Lucas (1978) Pricing Equation With R.E

Lets solve the stochastic difference equation (3) using rational expectations.

$$P_t = (\frac{1}{1+r})E_t[P_{t+1} + D_{t+1}] \tag{38}$$

The first iteration of the forward looking solution is:

$E_t[P_{t+1}] = (\frac{1}{1+r}))E_t[P_{t+2} + D_{t+2}]$

$P_t = (\frac{1}{1+r})[(\frac{1}{1+r})E_t[P_{t+2} + D_{t+2}]] + (\frac{1}{1+r})E_t[D_{t+1}] \Longleftrightarrow$

$\Longleftrightarrow P_t = (\frac{1}{1+r})^2 E_t[P_{t+2}] + (\frac{1}{1+r})^2[D_{t+2}] + (\frac{1}{1+r})E_t[D_{t+1}]$

Now moving to the second iteration:

$E_t[P_{t+2}] = (\frac{1}{1+r})E_t[P_{t+3} + D_{t+3}]$

$P_t = (\frac{1}{1+r})^2[(\frac{1}{1+r})E_t[P_{t+3} + D_{t+3}]] + (\frac{1}{1+r})^2[D_{t+2}] + (\frac{1}{1+r})E_t[D_{t+1}] \Longleftrightarrow$

$P_t = (\frac{1}{1+r})^3 E_t[P_{t+3}] + (\frac{1}{1+r})^3 E_t[D_{t+3}] + (\frac{1}{1+r})^2[D_{t+2}] + (\frac{1}{1+r})E_t[D_{t+1}]$

Now its possible to identify the solution's pattern:

$P_t = (\frac{1}{1+r})^N E_t[P_{t+N}] + \sum_{i=1}^{N}(\frac{1}{1+r})^i E_t[D_{t+i}]$

If one look for the long run solution ($i \to +\infty$), then:

$P_t = \sum_{i=1}^{N}(\frac{1}{1+r})^i E_t[D_{t+i}]$

The price will be strictly equal to the discounted dividends stream, as in Lucas (1978).

9.2.6 Blanchard (1979) Type Of Bubbles

The stochastic process defined by (9) respects the condition given by (7) and by consequence is also a solution for equation (3)

$E_t[B_{t+1}] = \pi \times E_t[\frac{1+r}{\pi}B_t + \epsilon_t] + (1-\pi)E_t[\epsilon_t] \Longleftrightarrow$

$E_t[B_{t+1}] = \pi\frac{1+r}{\pi}B_t + E_t[\epsilon_t] + (1-\pi) \times 0 \Longleftrightarrow$

$E_t[B_{t+1}] = (1+r)B_t$

9.2.7 Evans (1991) Type Of Bubbles

The stochastic process defined by (10) respects the condition given by (7) and by consequence is also a solution for equation (3)

If $B_t \leq \alpha$

$E_t[B_{t+1}] = E_t[(1+r)B_t \times u_t] = (1+r)B_t \times E_t[u_{t+1}] = (1+r)B_t$

If $B_t > \alpha$

$E_t[B_{t+1}] = \pi E_t[(\delta + \frac{(1+r)}{\pi} \times \underbrace{\theta_{t+1}}_{1} \times (B_t - \frac{\delta}{(1+r)})) \times u_{t+1}] + (1-\pi)E_t[\delta u_{t+1}] \Longleftrightarrow$

$E_t[B_{t+1}] = \pi(\delta + \frac{(1+r)}{\pi} \times (B_t - \frac{\delta}{(1+r)})) \times E_t[u_{t+1}] + (1-\pi)\delta \times E_t[u_{t+1}] \Longleftrightarrow$

$E_[B_{t+1}] = \pi\delta + (1+r)B_t - \delta + \delta - \pi\delta \Longleftrightarrow$

$E_t[B_{t+1}] = (1+r)B_t$