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# Relative numerical ranges

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## Abstract

Relying on the ideas of Stampfli [14] and Magajna [12] we introduce, for operators  $S$  and  $T$  on a separable complex Hilbert space, a new notion called the numerical range of  $S$  relative to  $T$  at  $r \in \sigma(|T|)$ . Some properties of these numerical ranges are proved. In particular, it is shown that the relative numerical ranges are non-empty convex subsets of the closure of the ordinary numerical range of  $S$ . We show that the position of zero with respect to the relative numerical range of  $S$  relative to  $T$  at  $\|T\|$  gives an information about the distance between the involved operators. This result has many interesting corollaries. For instance, one can characterize those complex numbers which are in the closure of the numerical range of  $S$  but are not in the spectrum of  $S$ .

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## 1. Introduction

Let  $\mathcal{H}$  be a separable complex Hilbert space. We denote by  $\mathcal{S}_{\mathcal{H}}$  the unit sphere of  $\mathcal{H}$  and by  $\mathcal{B}(\mathcal{H})$  the Banach algebra of all bounded linear operators

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on  $\mathcal{H}$ . The numerical range of  $S \in \mathcal{B}(\mathcal{H})$  is  $W(S) = \{\langle Sx, x \rangle; x \in \mathcal{S}_{\mathcal{H}}\}$ . It is well-known that  $W(S)$  is a non-empty convex subset of the disc  $\{z \in \mathbb{C}; |z| \leq \|S\|\}$ . The reader is referred to [8, 9, 10] for details about the numerical ranges. Many important properties of an operator are encoded in its numerical range. For instance,  $\sigma(S)$ , the spectrum of  $S$ , is a subset of  $\overline{W(S)}$ , the closure of the numerical range. In this paper, we are interested in some parts of  $\overline{W(S)}$  which are specified by an operator  $T \in \mathcal{B}(\mathcal{H})$ . We call them relative numerical ranges. They carry some useful information about the relation between  $S$  and  $T$ . We would like to point out that our study partially relies on the ideas of Stampfli [14] and Magajna [12].

The article is divided into two parts. In the first part, Section 2, we give a motivation for our definition of the relative numerical ranges and then we explore their properties. Among them the most important is convexity (Theorem 2.6). In the second part of the paper, Section 3, we show that the position of zero with respect to a relative numerical range gives an information about the distance between the involved operators. The main result of the section is Theorem 3.3 which extends a result proved by Stampfli [14] and therefore its proof is a modification of Stampfli's ideas, see also [6, 12]. Theorem 3.3 has several interesting corollaries, one of them is a characterization of the set  $\overline{W(S)} \setminus \sigma(S)$  for any  $S \in \mathcal{B}(\mathcal{H})$ .

## 2. Definition of the relative numerical ranges and their properties

To motivate our definition of the relative numerical ranges, we begin with a basic property of the ordinary numerical range: the numerical range of a compression of  $S$  to a closed subspace of  $\mathcal{H}$  is contained in  $W(S)$ . More precisely, if  $\mathcal{K}$  is a closed subspace of  $\mathcal{H}$  and  $P$  is the orthogonal projection onto  $\mathcal{K}$ , then  $W(PS|_{\mathcal{K}}) \subseteq W(S)$ . The following lemma gives a description of  $\overline{W(PS|_{\mathcal{K}})}$  which is the key idea in our definition of the relative numerical ranges.

**Lemma 2.1.** *Let  $\mathcal{K} \neq \{0\}$  be a closed subspace of  $\mathcal{H}$  and  $P$  be the orthogonal projection onto  $\mathcal{K}$ . Then, for every  $S \in \mathcal{B}(\mathcal{H})$ ,*

$$\overline{W(PS|_{\mathcal{K}})} = \left\{ \lambda \in \mathbb{C}; \exists (x_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\mathcal{H}} : \lim_{n \rightarrow \infty} \|Px_n\| = \|P\| \right. \\ \left. \text{and } \lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \lambda \right\}. \quad (2.1)$$

*Proof.* Assume that  $\lambda \in \overline{W(PS|_{\mathcal{X}})}$ . Then there exists a sequence  $(\lambda_n)_{n=1}^{\infty} \subseteq W(PS|_{\mathcal{X}})$  such that  $\lambda = \lim_{n \rightarrow \infty} \lambda_n$ . Hence there is a sequence  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\mathcal{X}}$  such that  $\lambda_n = \langle PS|_{\mathcal{X}} x_n, x_n \rangle$  for every  $n \in \mathbb{N}$ . Since  $Px_n = x_n$  and therefore  $\|Px_n\| = 1 = \|P\|$ , for every  $n$ , we conclude that  $\lim_{n \rightarrow \infty} \|Px_n\| = \|P\|$  and  $\lambda = \lim_{n \rightarrow \infty} \langle PS|_{\mathcal{X}} x_n, x_n \rangle = \lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle$ .

To prove the opposite inclusion, suppose that there exists a sequence  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\mathcal{X}}$  such that  $\lim_{n \rightarrow \infty} \|Px_n\| = \|P\|$  and  $\lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \lambda$ . Each vector  $x_n$  can be written as  $x_n = y_n + z_n$ , where  $y_n \in \mathcal{X}$  and  $y_n \in \mathcal{X}^{\perp}$ . Since  $\|y_n\|^2 + \|z_n\|^2 = 1$  and  $1 = \lim_{n \rightarrow \infty} \|Px_n\| = \lim_{n \rightarrow \infty} \|y_n\|$  we have  $\lim_{n \rightarrow \infty} \|z_n\| = 0$ . Without loss of generality we may assume that  $\frac{1}{2} < \|y_n\|$  for every  $n \in \mathbb{N}$ . Hence

$$\begin{aligned} & |\lambda - \langle S \frac{y_n}{\|y_n\|}, \frac{y_n}{\|y_n\|} \rangle| \\ & \leq |\lambda - \frac{1}{\|y_n\|^2} \langle Sx_n, x_n \rangle| + \frac{1}{\|y_n\|^2} |\langle Sy_n, z_n \rangle + \langle Sz_n, y_n \rangle + \langle Sz_n, z_n \rangle| \\ & \leq |\lambda - \langle Sx_n, x_n \rangle| + |\langle Sx_n, x_n \rangle - \frac{1}{\|y_n\|^2} \langle Sx_n, x_n \rangle| + 4 \cdot 3 \|S\| \|z_n\| \\ & \leq |\lambda - \langle Sx_n, x_n \rangle| + 4 \|S\| \|z_n\|^2 + 12 \|S\| \|z_n\| \\ & \leq |\lambda - \langle Sx_n, x_n \rangle| + 16 \|S\| \|z_n\|, \end{aligned}$$

which gives

$$\lim_{n \rightarrow \infty} |\lambda - \langle S \frac{y_n}{\|y_n\|}, \frac{y_n}{\|y_n\|} \rangle| \leq \lim_{n \rightarrow \infty} (|\lambda - \langle Sx_n, x_n \rangle| + 16 \|S\| \|z_n\|) = 0.$$

It is clear that  $(\frac{y_n}{\|y_n\|})_{n=1}^{\infty}$  is a sequence in  $\mathcal{S}_{\mathcal{X}}$  which means that  $\langle S \frac{y_n}{\|y_n\|}, \frac{y_n}{\|y_n\|} \rangle \in W(PS|_{\mathcal{X}})$  for every  $n \in \mathbb{N}$ . It follows that  $\lambda \in \overline{W(PS|_{\mathcal{X}})}$ .  $\square$

The set on the right hand side of (2.1) has meaning if  $P$  is replaced by an arbitrary  $T \in \mathcal{B}(\mathcal{H})$ . Let

$$\begin{aligned} W_T(S) = \{ \lambda \in \mathbb{C}; \quad & \exists (x_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\mathcal{X}} : \lim_{n \rightarrow \infty} \|Tx_n\| = \|T\| \\ & \text{and } \lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \lambda \}. \end{aligned} \quad (2.2)$$

Following Magajna [12], we call  $W_T(S)$  the numerical range of  $S$  relative to  $T$ . In the case  $S = T$ , (2.2) reduces to the Stampfli's maximal numerical range of  $T$ , see [14]. On the other hand,  $W_I(S) = W(S)$ , where  $I$  is the identity operator on  $\mathcal{H}$ . Actually, it is clear from the definition that  $W_T(S) = \overline{W(S)}$  for any operator  $T$  which is a scalar multiple of an isometry.

Recall that a number  $\lambda \in \mathbb{C}$  is an approximate eigenvalue of  $T \in \mathcal{B}(\mathcal{H})$  if there exists  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$ , called a sequence of approximate eigenvectors of  $T$  at  $\lambda$ , such that  $\lim_{n \rightarrow \infty} \|Tx_n - \lambda x_n\| = 0$ . It is obvious that the set  $\sigma_{ap}(T)$  of all approximate eigenvalues of  $T$  is a part of the spectrum  $\sigma(T)$  and it is well-known that  $\partial\sigma(T)$ , the boundary of  $\sigma(T)$ , is a subset of  $\sigma_{ap}(T)$ . In particular, if  $T$  is a selfadjoint operator, then  $\sigma(T) = \sigma_{ap}(T)$ .

Let  $|T|$  be the unique positive square root of  $T^*T$  and  $T = V|T|$  be the unique polar decomposition of  $T$  which satisfies the additional condition  $\ker(V) = \ker(|T|)$ , see [9, Problem 134]. Here  $V$  is a partial isometry whose initial space is  $\ker(V)^\perp$  and the final space is  $\text{im}(V) = \ker(V^*)^\perp$ , the initial space of  $V^*$ . It is not hard to see that  $\ker(T) = \ker(|T|)$  and  $\|Tx\| = \||T|x\|$  for every  $x \in \mathcal{H}$ . It follows that  $\|T\| = \||T|\|$  and  $m(T) = m(|T|)$ , where  $m(T) = \inf\{\|Tx\|; x \in \mathcal{S}_{\mathcal{H}}\}$  is the minimum modulus of  $T$ .

In the definition (2.2) of the numerical range of  $S$  relative to  $T$ , norming sequences of  $T$  are used, i.e.,  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  such that  $\lim_{n \rightarrow \infty} \|Tx_n\| = \|T\|$ . As the following lemma shows, the norming sequences of  $T$  are sequences of approximate eigenvectors of  $|T|$  at  $\|T\|$ .

**Lemma 2.2.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$ .*

(i) *If  $(x_n)_{n=1}^\infty$  is a norming sequence of  $T$ , then it is a sequence of approximate eigenvectors of  $|T|$  at  $\|T\|$ .*

(ii) *If  $\lim_{n \rightarrow \infty} \|Tx_n\| = m(T)$ , then  $(x_n)_{n=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $m(T)$ .*

(iii) *If  $(x_n)_{n=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $r \in \sigma(|T|)$ , then  $\lim_{n \rightarrow \infty} \|Tx_n\| = r$ .*

*Proof.* If  $T = 0$ , then  $|T| = 0$  and  $\|T\| = m(T) = 0$ . It is obvious that in this case claims (i)–(iii) hold for every sequence  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$ . Assume therefore that  $T \neq 0$ .

(i) Note that  $\|T\|^2 I - |T|^2$  is a positive operator. Let  $\sqrt{\|T\|^2 I - |T|^2}$  be its positive square root. If  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  is such that  $\lim_{n \rightarrow \infty} \|Tx_n\| = \|T\|$ , then

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\sqrt{\|T\|^2 I - |T|^2} x_n\|^2 &= \lim_{n \rightarrow \infty} \langle (\|T\|^2 I - |T|^2)x_n, x_n \rangle \\ &= \|T\|^2 - \lim_{n \rightarrow \infty} \|Tx_n\|^2 = 0. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \langle (\|T\|^2 I - |T|^2)x_n, x_n \rangle \leq \|\sqrt{\|T\|^2 I - |T|^2}\| \lim_{n \rightarrow \infty} \|\sqrt{\|T\|^2 I - |T|^2} x_n\| = 0.$$

Since  $\|T\|I + |T|$  is an invertible operator and

$$\|(\|T\|I - |T|)x_n\| \leq \|(\|T\|I + |T|)^{-1}\| \|(\|T\|^2I - |T|^2)x_n\|$$

holds for every  $x_n$  we conclude that  $\lim_{n \rightarrow \infty} \| |T|x_n - \|T\|x_n \| = 0$ . Hence,  $(x_n)_{n=1}^{\infty}$  is a sequence of approximate eigenvectors for  $|T|$  at  $\|T\|$ .

(ii) The proof of this assertion is similar to the proof of (i); one has to observe that now  $|T|^2 - m(T)^2I$  is a positive semidefinite operator.

(iii) Assume that  $(x_n)_{n=1}^{\infty}$  is a sequence of approximate eigenvectors for  $|T|$  at  $r$ . Since  $|\|Tx_n\| - r| = | \| |T|x_n \| - \|rx_n \| | \leq \| |T|x_n - rx_n \|$  holds for every  $x_n$ , we have  $\lim_{n \rightarrow \infty} \|Tx_n\| = r$ .  $\square$

**Remark 2.3.** The opposite implication in Lemma 2.2 (iii), i.e., a statement similar to (i) and (ii), does not hold in general for  $r \in \sigma(|T|)$  if  $r \neq \|T\|$  and  $r \neq m(T)$ . For instance, let  $T \in \mathcal{B}(\mathcal{H})$  be an operator such that  $m(T) < \|T\|$ . Then the closure of the numerical range of  $T^*T = |T|^2$  is  $\overline{W(|T|^2)} = [m(T)^2, \|T\|^2]$ . Hence, for every  $t \in (m(T), \|T\|)$ , there exists a vector  $x_t \in \mathcal{S}_{\mathcal{H}}$  such that  $\langle |T|^2x_t, x_t \rangle = t^2$  which gives  $\|Tx_t\| = t$ . Of course, it is not necessary that  $|T|x_t = tx_t$  ( $t$  even does not need to be in the spectrum of  $|T|$ ). Let us consider a more explicit example of a  $3 \times 3$  diagonal matrix  $T = \text{diag}[1, r, s]$  where  $0 \leq s < r < 1$ . Hence  $T = |T|$ ,  $\|T\| = 1$ , and  $m(T) = s$ . For every  $t \in (s, 1)$ , let  $x_t = \left( \sqrt{\frac{t^2 - s^2}{1 - s^2}}, 0, \sqrt{\frac{1 - t^2}{1 - s^2}} \right)$ . Then  $\|x_t\| = 1$ ,  $\|Tx_t\| = t$  but  $Tx_t \neq tx_t$ . Note that even in the case  $t = r$  the vector  $x_r$  is not an eigenvector of  $T$  at  $r$ .

It follows from Lemma 2.2 that in the definition (2.2) sequences of norming vectors of  $T$  can be replaced by sequences of approximate eigenvectors of  $T$  at  $\|T\|$ . Since each number in  $\sigma(|T|)$  is an approximate eigenvalue of  $|T|$  we extend the definition (2.2) as follows.

**Definition 2.4.** Let  $T \in \mathcal{B}(\mathcal{H})$  and  $r \in \sigma(|T|)$ . The numerical range of  $S \in \mathcal{B}(\mathcal{H})$  relative to  $T$  at  $r$  is

$$W_T^r(S) = \{ \lambda \in \mathbb{C}; \exists (x_n)_{n=1}^{\infty} \subseteq \mathcal{S}_{\mathcal{H}} : \lim_{n \rightarrow \infty} \| |T|x_n - rx_n \| = 0 \text{ and } \lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \lambda \}. \quad (2.3)$$

Note that it follows from the definition that  $W_T^r(S) = W_{|T|}^r(S)$ ; for a slightly more general assertion see Proposition 2.7 (ii).

**Lemma 2.5.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and  $r \in \sigma(|T|)$ . Then  $W_T^r(S)$  is a non-empty closed subset of  $\overline{W(S)}$  for every  $S \in \mathcal{B}(\mathcal{H})$ .*

*Proof.* Let  $S \in \mathcal{B}(\mathcal{H})$  be an arbitrary operator. For every sequence  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  of approximate eigenvectors of  $|T|$  at  $r$ , the sequence  $(\langle Sx_n, x_n \rangle)_{n=1}^\infty$  is bounded which means that there exists a subsequence  $(x_{n_k})_{k=1}^\infty$  such that  $(\langle Sx_{n_k}, x_{n_k} \rangle)_{k=1}^\infty$  converges to a number  $\lambda$ . It is obvious that  $(x_{n_k})_{k=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $r$  which means that  $\lambda \in W_T^r(S)$ , i.e.,  $W_T^r(S)$  is a non-empty set. The inclusion  $W_T^r(S) \subseteq \overline{W(S)}$  is trivial.

Assume that  $\lambda \in \overline{W_T^r(S)}$  and that  $(\lambda_k)_{k=1}^\infty \subseteq W_T^r(S)$  is a sequence which converges to  $\lambda$ . Without loss of generality we may assume that  $|\lambda - \lambda_k| < \frac{1}{k}$ . For each  $k \in \mathbb{N}$ , let  $(x_n^{(k)})_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  be a sequence of approximate eigenvectors of  $|T|$  at  $r$  such that  $\lambda_k = \lim_{n \rightarrow \infty} \langle Sx_n^{(k)}, x_n^{(k)} \rangle$ . It is clear that for each  $k$  there exists an index  $n_k$  such that

$$\| |T|x_{n_k}^{(k)} - rx_{n_k}^{(k)} \| < \frac{1}{k} \quad \text{and} \quad |\lambda_k - \langle Sx_{n_k}^{(k)}, x_{n_k}^{(k)} \rangle| < \frac{1}{k}.$$

Hence  $(x_{n_k}^{(k)})_{k=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $r$  such that

$$|\lambda - \langle Sx_{n_k}^{(k)}, x_{n_k}^{(k)} \rangle| \leq |\lambda - \lambda_k| + |\lambda_k - \langle Sx_{n_k}^{(k)}, x_{n_k}^{(k)} \rangle| < \frac{2}{k}.$$

We conclude that  $\lambda \in W_T^r(S)$ . □

One of the basic and probably the most important property of the ordinary numerical range is its convexity. Modifying the proof of [6, Lemma 9.13] (see also [12, 14]) we are able to show that the relative numerical ranges are convex sets, as well.

**Theorem 2.6.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and  $r \in \sigma(|T|)$ . Then  $W_T^r(S)$  is a convex set for every  $S \in \mathcal{B}(\mathcal{H})$ .*

*Proof.* Without loss of generality we may assume that  $\|S\| = \|T\| = 1$ . Note that it follows from this that  $0 \leq r \leq 1$ . We have to prove that for arbitrary  $\lambda, \mu \in W_T^r(S)$  the line segment  $[\lambda, \mu]$  is a subset of  $W_T^r(S)$ . If  $\lambda = \mu$ , then there is nothing to prove. Assume therefore that  $\lambda \neq \mu$ . By the definition, there exist sequences of approximating eigenvectors  $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  of  $|T|$  at  $r$ , i.e.,

$$\lim_{n \rightarrow \infty} \| |T|x_n - rx_n \| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \| |T|y_n - ry_n \| = 0, \quad (2.4)$$

such that

$$\lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \lambda \quad \text{and} \quad \lim_{n \rightarrow \infty} \langle Sy_n, y_n \rangle = \mu. \quad (2.5)$$

If we replace  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  by their subsequences, then (2.5) still hold, however (2.4) can be replaced by

$$\| |T|x_n - rx_n \| < \frac{1}{n} \quad \text{and} \quad \| |T|y_n - ry_n \| < \frac{1}{n} \quad (\forall n \in \mathbb{N}). \quad (2.6)$$

Thus, it is assumed now on that  $(x_n)_{n=1}^{\infty}$  and  $(y_n)_{n=1}^{\infty}$  satisfy conditions (2.5) and (2.6).

Denote  $\lambda_n = \langle Sx_n, x_n \rangle$  and  $\mu_n = \langle Sy_n, y_n \rangle$ . Since  $\lambda \neq \mu$  and  $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ ,  $\lim_{n \rightarrow \infty} \mu_n = \mu$  it is obvious that there exists an index  $n_0$  such that  $|\mu_n - \lambda_n| \geq \frac{1}{2}|\mu - \lambda|$  for all  $n \geq n_0$ . We may assume that  $n_0 = 1$ . On the other hand, note that  $\lambda_n, \mu_n$  are numbers in  $W(S)$  which gives  $|\mu_n - \lambda_n| \leq |\mu_n| + |\lambda_n| \leq 2\|S\| = 2$ . Similarly,  $|\mu - \lambda| \leq 2$ .

For each  $n \in \mathbb{N}$ , let  $\epsilon_n = \frac{\langle x_n, y_n \rangle}{|\langle x_n, y_n \rangle|}$  if  $\langle x_n, y_n \rangle \neq 0$  and  $\epsilon_n = 0$  otherwise. Denote  $v_n = \epsilon_n y_n - x_n$ . Then

$$\mu_n = \langle S(\epsilon_n y_n), \epsilon_n y_n \rangle = \langle S(x_n + v_n), x_n + v_n \rangle = \lambda_n + \langle Sx_n, v_n \rangle + \langle Sv_n, x_n + v_n \rangle,$$

and therefore

$$|\mu_n - \lambda_n| = |\langle Sx_n, v_n \rangle + \langle Sv_n, x_n + v_n \rangle| \leq |\langle Sx_n, v_n \rangle| + |\langle Sv_n, \epsilon_n y_n \rangle| \leq 2\|v_n\|.$$

Since

$$1 - \frac{1}{2}\|v_n\|^2 = 1 - \frac{1}{2}\langle \epsilon_n y_n - x_n, \epsilon_n y_n - x_n \rangle = 1 - \frac{1}{2}(2 - 2\operatorname{Re}(\langle x_n, \epsilon_n y_n \rangle)) = |\langle x_n, y_n \rangle|$$

we have

$$|\langle x_n, y_n \rangle| = 1 - \frac{1}{2}\|v_n\|^2 \leq 1 - \frac{1}{8}|\mu_n - \lambda_n|^2 \leq 1 - \frac{1}{32}|\mu - \lambda|^2 < 1$$

for every  $n \in \mathbb{N}$ . It follows from

$$\begin{aligned} | \|T^*Tx_n\|^2 - r^4 | &= (\|T^*Tx_n\| + r^2) | \|T^*Tx_n\| - r^2 | \\ &\leq 2 | \| |T|^2x_n \| - \|r^2x_n \| | \leq \| (|T|^2 - r^2I)x_n \| \\ &\leq 2 \| |T| + rI \| \| |T|x_n - rx_n \| < \frac{4}{n} \end{aligned}$$



that

$$\begin{aligned}\|(r^2I - T^*T)x_n\|^2 &= r^4 + \|T^*Tx_n\|^2 - 2r^2\|Tx_n\|^2 \leq r^4 + r^4 + \frac{4}{n} - 2r^2\|Tx_n\|^2 \\ &\leq 2r^2(\|Tx_n\| + r) \|\|Tx_n\| - r\| + \frac{4}{n} \leq 4\|Tx_n - rx_n\| + \frac{4}{n} < \frac{8}{n}.\end{aligned}$$

Similarly,  $\|(r^2I - T^*T)y_n\|^2 < \frac{8}{n}$ .

Let  $\mathcal{V}\{x_n, y_n\}$  be the subspace of  $\mathcal{H}$  which is spanned by  $x_n$  and  $y_n$ . If  $u_n \in \mathcal{V}\{x_n, y_n\}$  is a unit vector and  $u_n = a_nx_n + b_ny_n$ , where  $a_n, b_n \in \mathbb{C}$ , then

$$\begin{aligned}1 = \|u_n\|^2 &= \langle a_nx_n + b_ny_n, a_nx_n + b_ny_n \rangle = |a_n|^2 + |b_n|^2 + 2\operatorname{Re}(a_n\bar{b}_n\langle x_n, y_n \rangle) \\ &\geq |a_n|^2 + |b_n|^2 - 2|a_n||b_n|\langle x_n, y_n \rangle \\ &\geq |a_n|^2 + |b_n|^2 - 2|a_n||b_n|(1 - \frac{1}{32}|\mu - \lambda|) \\ &= \frac{1}{32}|\mu - \lambda|(|a_n|^2 + |b_n|^2) + (1 - \frac{1}{32}|\mu - \lambda|)(|a_n| - |b_n|)^2 \\ &\geq \frac{1}{32}|\mu - \lambda|(|a_n|^2 + |b_n|^2).\end{aligned}$$

It follows that  $|a_n|^2 + |b_n|^2 \leq 32/|\mu - \lambda|$ . In the following we use first the triangle inequality and after the Cauchy-Schwartz inequality:

$$\begin{aligned}\|(r^2I - T^*T)u_n\| &\leq |a_n|\|(r^2I - T^*T)x_n\| + |b_n|\|(r^2I - T^*T)y_n\| \\ &\leq \sqrt{|a_n|^2 + |b_n|^2}\sqrt{\|(r^2I - T^*T)x_n\|^2 + \|(r^2I - T^*T)y_n\|^2} \\ &\leq \frac{4\sqrt{2}}{\sqrt{|\mu - \lambda|}}\sqrt{\frac{8}{n} + \frac{8}{n}} = \frac{16\sqrt{2}}{\sqrt{|\mu - \lambda|}} \cdot \frac{1}{\sqrt{n}}.\end{aligned}$$

Hence

$$\begin{aligned}\|ru_n - |T|u_n\| &= \|(rI + |T|)^{-1}(r^2I - |T|^2)u_n\| \\ &\leq \|(rI + |T|)^{-1}\|\|(r^2I - T^*T)u_n\| \\ &\leq \frac{16\sqrt{2}\|(rI + |T|)^{-1}\|}{\sqrt{|\mu - \lambda|}} \cdot \frac{1}{\sqrt{n}},\end{aligned}$$

which gives  $\lim_{n \rightarrow \infty} \|ru_n - |T|u_n\| = 0$ , that is,  $(u_n)_{n=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $r$ .

Let  $\eta = t\lambda + (1-t)\mu$  for some  $t \in [0, 1]$ . Then  $\eta_n = t\lambda_n + (1-t)\mu_n$  ( $n \in \mathbb{N}$ ) is a sequence which converges to  $\eta$ . Let  $P_n \in \mathcal{B}(\mathcal{H})$  be the orthogonal projection onto  $\mathcal{V}\{x_n, y_n\}$  along  $\mathcal{V}\{x_n, y_n\}^\perp$ . Denote by  $Q_n$  the compression of  $S$  to  $\mathcal{V}\{x_n, y_n\}$ , i.e.,  $Q_n = P_nS|_{\mathcal{V}\{x_n, y_n\}}$ . Then  $\lambda_n = \langle Sx_n, x_n \rangle = \langle Q_nx_n, x_n \rangle \in W(Q_n)$  and, similarly,  $\mu_n \in W(Q_n)$ . Because of the convexity of  $W(Q_n)$  one

has  $\eta_n \in W(Q_n)$  which means that there is a unit vector  $u_n \in \bigvee\{x_n, y_n\}$  such that  $\langle Su_n, u_n \rangle = \langle Q_n u_n, u_n \rangle = \eta_n$ . By the previous paragraph,  $(u_n)_{n=1}^\infty$  is a sequence of approximate eigenvectors of  $|T|$  at  $r$ . We conclude that  $\eta = \lim_{n \rightarrow \infty} \eta_n = \lim_{n \rightarrow \infty} \langle Su_n, u_n \rangle \in W_T^r(S)$ .  $\square$

Now we list a few more properties of the relative numerical ranges.

**Proposition 2.7.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and  $r \in \sigma(|T|)$ .*

(i) *For every  $S \in \mathcal{B}(\mathcal{H})$ , one has  $W_T^r(S^*) = \{\bar{\lambda}; \lambda \in W_T^r(S)\}$  and  $W_T^r(\alpha S + \beta I) = \alpha W_T^r(S) + \beta$ , where  $\alpha, \beta \in \mathbb{C}$  are arbitrary.*

(ii) *Assume that  $\gamma \in \mathbb{C}$  is a non-zero number and  $V \in \mathcal{B}(\mathcal{H})$  is an isometry. Then  $W_{\gamma VT}^{|\gamma|r}(S) = W_T^r(S)$  holds for every  $S \in \mathcal{B}(\mathcal{H})$ .*

(iii) *If  $f$  is a continuous real-valued function on  $\sigma(|T|)$ , then  $W_T^r(S) \subseteq W_{f(|T|)}^{f(r)}(S)$  for every  $S \in \mathcal{B}(\mathcal{H})$ . Moreover, if  $f$  is injective and  $f(t) \geq 0$  for all  $t \in \sigma(|T|)$ , then  $W_T^r(S) = W_{f(|T|)}^{f(r)}(S)$  for every  $S \in \mathcal{B}(\mathcal{H})$ .*

*Proof.* (i) is obvious. For (ii) observe that  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  is a sequence of approximate eigenvectors of  $|T|$  at  $r$  if and only if it is a sequence of approximate eigenvectors of  $|\alpha VT| = |\alpha||T|$  at  $|\alpha|r$ . To prove (iii), assume that  $\lambda \in W_T^r(S)$  and let  $(x_n) \subseteq \mathcal{S}_{\mathcal{H}}$  be a sequence of approximate eigenvectors of  $|T|$  at  $r$  such that  $\lambda = \lim_{n \rightarrow \infty} \langle S^* T x_n, x_n \rangle$ . Since  $f$  is continuous one has  $\lim_{n \rightarrow \infty} \|f(|T|)x_n - f(r)x_n\| = 0$  and therefore  $\lim_{n \rightarrow \infty} \|f(|T|)|x_n - |f(r)|x_n\| = 0$ . Hence  $(x_n) \subseteq \mathcal{S}_{\mathcal{H}}$  is a sequence of approximate eigenvectors of  $|f(|T|)|$  at  $|f(r)|$  and therefore  $\lambda \in W_{f(|T|)}^{f(r)}(S)$ . If  $f$  is injective and  $f(t) \geq 0$  for all  $t \in \sigma(|T|)$ , then it is a bijection from  $\sigma(|T|)$  to  $\sigma(f(|T|))$  and its inverse  $f^{-1} : \sigma(f(|T|)) \rightarrow \sigma(|T|)$  is continuous. By the previous part of the proof,  $W_T^r(S) = W_{f^{-1}(f(|T|))}^{f^{-1}(f(r))}(S) \supseteq W_{f(|T|)}^{f(r)}(S)$  for every  $S \in \mathcal{B}(\mathcal{H})$ .  $\square$

**Proposition 2.8.** *Let  $T \in \mathcal{B}(\mathcal{H})$  and  $r \in \sigma(|T|)$ . If  $U \in \mathcal{B}(\mathcal{H})$  is a unitary operator, then  $W_{TU}^r(U^* S U) = W_T^r(S)$  for every  $S \in \mathcal{B}(\mathcal{H})$ . Moreover, if  $U$  and  $|T|$  commute, then  $W_T^r(U^* S U) = W_T^r(S)$ .*

*Proof.* Note first that  $|TU|^2 = (U^* |T| U)^2$  which gives  $|TU| = U^* |T| U$  as  $|TU|$  is the unique positive square root of  $U^* T^* T U$ . Hence  $\sigma(|T|) = \sigma(|TU|)$ . Let  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  be a sequence of approximate eigenvectors of  $|T|$  at  $r$  such that  $\lambda = \lim_{n \rightarrow \infty} \langle S x_n, x_n \rangle$ . Then  $(U^* x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  is a sequence of approximate eigenvectors of  $|TU|$  at  $r$  such that  $\lim_{n \rightarrow \infty} \langle U^* S U U^* x_n, U^* x_n \rangle = \lambda$ .

This proves the inclusion  $W_T^r(S) \subseteq W_{TU}^r(U^*SU)$ . The opposite inclusion follows from this one by replacing  $T$  with  $TU$  and  $U$  with  $U^*$ . If  $|T|$  and  $U$  commute, then  $|TU| = U^*|T|U = |T|$  and therefore  $W_T^r(S) = W_{TU}^r(U^*SU) = W_{|TU|}^r(U^*SU) = W_{|T|}^r(U^*SU) = W_T^r(U^*SU)$ .  $\square$

**Corollary 2.9.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be an invertible operator with the polar decomposition  $T = U|T|$  and let  $r \in \sigma(|T|)$ . Then  $W_{T^*}^r(S) = W_T^r(U^*SU)$  for every  $S \in \mathcal{B}(\mathcal{H})$ .*

*Proof.* Since  $T$  is invertible  $U$  is unitary. It follows from  $|T^*|^2 = (U|T|U^*)^2$  that  $|T^*| = U|T|U^*$ . Hence  $W_{T^*}^r(S) = W_{|T^*|}^r(S) = W_{U|T|U^*}^r(S) = W_{|T|U^*}^r(S)$ , where the last equality holds because of Proposition 2.7 (ii). Now, by Proposition 2.8, we conclude that  $W_{|T|U^*}^r(S) = W_{|T|}^r(U^*SU) = W_T^r(U^*SU)$ .  $\square$

**Remark 2.10.** In general the relative numerical ranges are not invariant under unitary equivalence, i.e.,  $W_T^r(S)$  and  $W_T^r(U^*SU)$  can be different. On the other hand, commutativity of  $U$  and  $|T|$  which we used in Proposition 2.8 to obtain  $W_T^r(U^*SU) = W_T^r(S)$  for every  $S \in \mathcal{B}(\mathcal{H})$  is not necessary. Consider the following example. Let  $T = \text{diag}[1, r, s]$ , where  $0 \leq s < r < 1$ , and let  $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ . If  $x_n = (\alpha_n, \beta_n, \gamma_n) \in \mathcal{S}_{\mathbb{C}^3}$  ( $n \in \mathbb{N}$ ) is a sequence of approximating eigenvectors of  $T$  at 1, then  $|\alpha_n| \rightarrow 1$ ,  $|\beta_n| \rightarrow 0$  and  $|\gamma_n| \rightarrow 0$  as  $n \rightarrow \infty$ . It follows, for every  $3 \times 3$  matrix  $S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$ , that  $\lim_{n \rightarrow \infty} \langle Sx_n, x_n \rangle = \sigma_{11}$  and  $\lim_{n \rightarrow \infty} \langle U^*SUX_n, x_n \rangle = \sigma_{11}$  which gives  $W_T^1(S) = \{\sigma_{11}\} = W_T^1(U^*SU)$ . On the other hand, if  $y_n = (\delta_n, \epsilon_n, \xi_n) \in \mathcal{S}_{\mathbb{C}^3}$  ( $n \in \mathbb{N}$ ) is a sequence of approximating eigenvectors of  $T$  at  $r$ , then  $|\delta_n| \rightarrow 0$ ,  $|\epsilon_n| \rightarrow 1$  and  $|\xi_n| \rightarrow 0$  as  $n \rightarrow \infty$ . Hence  $\lim_{n \rightarrow \infty} \langle Sy_n, y_n \rangle = \sigma_{22}$  and  $\lim_{n \rightarrow \infty} \langle U^*SUY_n, y_n \rangle = \sigma_{33}$  which gives  $W_T^1(S) \neq W_T^1(U^*SU)$  if  $\sigma_{22} \neq \sigma_{33}$ .

### 3. Zero in the relative numerical range

It is known that the position of zero with respect to the numerical range of  $S \in \mathcal{B}(\mathcal{H})$  gives some information about  $S$ . In this section we show that the presence of 0 in  $W_T^r(S^*T)$  gives lower bound for the distance from  $T$  to the linear space spanned by  $S$ . The converse holds as well in the sense that, for given  $S, T \in \mathcal{B}(\mathcal{H})$ , zero is in  $W_T^r(S^*T)$  if  $T$  is at the maximal possible distance to  $\overline{\mathbb{C}S}$ . Using this and related results we can characterize numbers which are in  $\overline{W(S)} \setminus \sigma(T)$  (see Theorem 3.14).

**Proposition 3.1.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a non-zero operator and  $r \in \sigma(|T|)$ . If  $0 \in W_T^r(S^*T)$ , for  $S \in \mathcal{B}(\mathcal{H})$ , then  $\text{dist}(T, \mathbb{C}S) \geq r$ . Hence,  $\text{dist}(T, \mathbb{C}S) \geq \sup\{r \in \sigma(|T|); 0 \in W_T^r(S^*T)\}$ .*

*Proof.* Let  $S \in \mathcal{B}(\mathcal{H})$  be an arbitrary operator such that  $0 \in W_T^r(S^*T)$ . Then there exists a sequence  $(x_n)_{n=1}^\infty \subseteq \mathcal{S}_{\mathcal{H}}$  of approximate eigenvectors of  $|T|$  at  $r$  such that  $\lim_{n \rightarrow \infty} \langle S^*T x_n, x_n \rangle = 0$ . It follows that  $\lim_{n \rightarrow \infty} \text{Re}(\bar{\lambda} \langle S^*T x_n, x_n \rangle) = 0$  for every  $\lambda \in \mathbb{C}$ . Let now  $\lambda$  be arbitrary but fixed. Since

$$\|(T - \lambda S)x_n\|^2 - |\lambda|^2 \|Sx_n\|^2 = \|Tx_n\|^2 - 2\text{Re}(\bar{\lambda} \langle S^*T x_n, x_n \rangle) \quad (\forall n \in \mathbb{N})$$

and  $\lim_{n \rightarrow \infty} \|Tx_n\|^2 = r^2$ , by Lemma 2.2, for any  $\varepsilon > 0$  there exists an index  $n_\varepsilon$  such that

$$r^2 - \varepsilon < \|(T - \lambda S)x_n\|^2 - |\lambda|^2 \|Sx_n\|^2 < r^2 + \varepsilon \quad (\forall n \geq n_\varepsilon).$$

It follows that  $r^2 \leq r^2 + |\lambda|^2 \|Sx_n\|^2 \leq \|(T - \lambda S)x_n\|^2 + \varepsilon \leq \|T - \lambda S\|^2 + \varepsilon$  and we may conclude that  $r \leq \|T - \lambda S\|$ . Since  $\lambda$  is an arbitrary number the assertion follows.  $\square$

**Remark 3.2.** If  $r < \|T\|$ , in Proposition 3.1, then the inequality  $\text{dist}(T, \mathbb{C}S) \geq r$  can be strict. For instance, if  $r_1, r_2 \in \sigma(|T|)$  are such that  $r_1 < r_2$  and  $0 \in W_T^{r_i}(S^*T)$  ( $i = 1, 2$ ), then  $\text{dist}(T, \mathbb{C}S) > r_1$ . For an explicit example consider again the  $3 \times 3$  matrix  $T = \text{diag}[1, r, s]$  where  $0 \leq s < r < 1$ . Let  $S = \text{diag}[0, 0, 1]$ . It is easily seen that  $e_1 = (1, 0, 0)$  is an eigenvector of  $T$  at the eigenvalue 1 (which is the norm of  $T$ ) and  $e_2 = (0, 1, 0)$  is an eigenvector of  $T$  at  $r$ . Since  $S^*T e_i = 0$  ( $i = 1, 2$ ) we have  $0 \in W_T^{r_i}(S^*T)$  and  $0 \in W_T^1(S^*T)$ . It follows that  $\text{dist}(T, \mathbb{C}S) = 1 > r$ .

For  $r = \|T\|$  Proposition 3.1 implies  $\text{dist}(T, \mathbb{C}S) = \|T\|$  whenever  $0 \in W_T^{\|T\|}(S^*T)$  because the inequality  $\text{dist}(T, \mathbb{C}S) \leq \|T\|$  always holds. Actually it can be proved that conditions  $\text{dist}(T, \mathbb{C}S) = \|T\|$  and  $0 \in W_T^{\|T\|}(S^*T)$  are equivalent. Bhatia and Šemrl formulated this as a result about Birkhoff-James orthogonality between matrices and operators, see [2, Theorem 1.1, Remark 3.1]. However we will modify the proof of [6, Lemma 9.14] (see also [12, 14]) and prove the following theorem.

**Theorem 3.3.** *Let  $S, T \in \mathcal{B}(\mathcal{H})$  be arbitrary. Then  $\|T\| = \text{dist}(T, \mathbb{C}S)$  if and only if  $0 \in W_T^{\|T\|}(S^*T)$ .*

*Proof.* Note that the assertion trivially holds for  $T = 0$ . Assume therefore that  $T \neq 0$ . We have already observed that one implication follows from Proposition 3.1. To prove the opposite implication, suppose that  $0 \notin W_T^{\|T\|}(S^*T)$ . Since  $W_T^{\|T\|}(\alpha S^*T) = \alpha W_T^{\|T\|}(S^*T)$  and  $\|T\| = \text{dist}(T, \mathbb{C}S)$  if and only if  $\|T\| = \text{dist}(T, \mathbb{C}(\alpha S))$  for any  $\alpha \neq 0$ , we can assume that  $\|S\| = 1$ . Further, because of the convexity of  $W_T^{\|T\|}(S^*T)$  we may replace  $T$  by its scalar multiple (if necessary) to get  $\text{Re}(W_T^{\|T\|}(S^*T)) \geq 1$ . However, since  $W_T^{\|T\|}(S^*T) \subseteq \overline{W(S^*T)}$  it is still possible that there are numbers in  $\overline{W(S^*T)}$  with real part smaller than 1. Let  $\mathcal{M} = \{x \in \mathcal{S}_{\mathcal{H}}; \text{Re}(\langle S^*Tx, x \rangle) \leq \frac{1}{2}\}$  and let  $\rho = \sup\{\|Tx\|; x \in \mathcal{M}\}$ . It is obvious that  $\rho \leq \|T\|$ . Moreover, one actually has  $\rho < \|T\|$ . Indeed, if  $\rho = \|T\|$ , then there would exist a sequence  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{M}$  such that  $\lim_{n \rightarrow \infty} \|Tx_n\| = \|T\|$ . Since the sequence of numbers  $(\langle S^*Tx_n, x_n \rangle)_{n=1}^{\infty}$  is bounded it has a convergent subsequence, say  $\eta = \lim_{k \rightarrow \infty} \langle S^*Tx_{n_k}, x_{n_k} \rangle$ . Because of  $\lim_{k \rightarrow \infty} \|Tx_{n_k}\| = \|T\|$  one has  $\eta \in W_T^{\|T\|}(S^*T)$  and therefore  $\text{Re}(\eta) \geq 1$ . However, on the other hand, it should be also  $\text{Re}(\eta) \leq \frac{1}{2}$ . Hence  $\rho < \|T\|$ . Let  $\mu = \min\{\frac{1}{2}, \frac{1}{2}(\|T\| - \rho)\}$ . It is obvious that  $0 < \mu < 1$ . Consider  $T - \mu S$ . If  $x \in \mathcal{M}$ , then

$$\|(T - \mu S)x\| \leq \|Tx\| + \mu\|Sx\| \leq \rho + \frac{1}{2}(\|T\| - \rho) < \|T\|. \quad (3.1)$$

For  $x \in \mathcal{S}_{\mathcal{H}} \setminus \mathcal{M}$ , one has  $\text{Re}(\langle S^*Tx, x \rangle) > \frac{1}{2}$  and therefore

$$\|(T - \mu S)x\|^2 = \|Tx\|^2 + \mu^2\|Sx\|^2 - 2\mu \text{Re}(\langle S^*Tx, x \rangle) < \|T\|^2 - \mu(1 - \mu). \quad (3.2)$$

It follows from (3.1) and (3.2) that  $\|T - \mu S\| < \|T\|$ , i.e.,  $\text{dist}(T, \mathbb{C}S) < \|T\|$ .  $\square$

**Corollary 3.4.** *Let  $V \in \mathcal{B}(\mathcal{H})$  be an isometry and  $S \in \mathcal{B}(\mathcal{H})$  be an arbitrary operator. Then  $\text{dist}(V, \mathbb{C}S) = 1$  if and only if  $0 \in \overline{W(V^*S)}$ . In particular,  $\text{dist}(I, \mathbb{C}S) = 1$  if and only if  $0 \in \overline{W(S)}$ .*

*Proof.* Let  $T = V$  in Theorem 3.3. Since  $W_V^1(S^*V) = \overline{W(S^*V)}$  and since  $0 \in \overline{W(S^*V)}$  if and only if  $0 \in \overline{W(V^*S)}$  the assertion follows.  $\square$

**Corollary 3.5.** *Let  $V \in \mathcal{B}(\mathcal{H})$  be an isometry and  $P \in \mathcal{B}(\mathcal{H})$  be an invertible positive operator. Then  $\text{dist}(V, \mathbb{C}VP) < 1$ .*

*Proof.* Since  $0 \notin \sigma(P) = \overline{W(P)} = \overline{W(V^*(VP))}$  the assertion follows, by Corollary 3.4.  $\square$

It is well known that the set  $\mathcal{B}^{-1}$  of all invertible operators in  $\mathcal{B}(\mathcal{H})$  contains the open ball of radius  $\frac{1}{\|A^{-1}\|}$  about each  $A \in \mathcal{B}^{-1}$ . In particular, the open unit ball about a unitary operator is contained in  $\mathcal{B}^{-1}$ . Since a non-zero scalar multiple of an invertible operator is invertible, as well, it follows from next corollary that  $\mathcal{B}^{-1}$  is determined by the union of unit balls centered at unitary operators in the sense that  $S \in \mathcal{B}(\mathcal{H})$  is invertible if and only if there exists a unitary operator  $U$  and a scalar multiple of  $S$  which is contained in the open unit ball about  $U$ .

**Corollary 3.6.** *An operator  $S \in \mathcal{B}(\mathcal{H})$  is non-invertible if and only if  $\text{dist}(U, \mathbb{C}S) = 1$  for every unitary operator  $U$ .*

*Proof.* By [7, Proposition 3.3],  $S \in \mathcal{B}(\mathcal{H})$  is not invertible if and only if  $0 \in \overline{W(U^*S)}$  for every unitary operator  $U$ . But this is equivalent, by Corollary 3.4, to  $\text{dist}(U, \mathbb{C}S) = 1$  for any unitary operator  $U$ .  $\square$

**Corollary 3.7.** *If  $S \in \mathcal{B}^{-1}$ , then there exists a unitary operator  $U \in \mathcal{B}(\mathcal{H})$  and a (non-zero) number  $\alpha$  such that  $\|U^* - \alpha S^{-1}\| < 1$ . In particular, if  $\|I - S\| < 1$ , then there exists  $\alpha \in \mathbb{C} \setminus \{0\}$  such that  $\|I - \alpha S^{-1}\| < 1$ .*

*Proof.* Since  $S$  is invertible, by [7, Proposition 3.3], there exists a unitary operator  $U$  such that  $0 \notin \overline{W(U^*S)}$ . By [5, Theorem 3.6],  $0 \notin \overline{W(S^{-1}U)}$ . Since the numerical ranges are invariant under unitary equivalences one has  $0 \notin \overline{W(US^{-1})}$ . Hence, by Corollary 3.4,  $\text{dist}(U^*, \mathbb{C}S^{-1}) < 1$ , i.e., there exists  $\alpha \neq 0$  such that  $\|U^* - \alpha S^{-1}\| < 1$ . For the second part observe that  $\|I - S\| < 1$  gives invertibility of  $S$ .  $\square$

**Corollary 3.8.** *If  $T \in \mathcal{B}(\mathcal{H})$  is invertible, then there exists a number  $\lambda$  such that  $\|T^* - \lambda T^{-1}\| < 1$ .*

*Proof.* Since  $0 \notin W_T^{\|T\|}(T^{-1}T)$  there exists a number  $\bar{\lambda}$  such that  $\|T - \bar{\lambda}(T^{-1})^*\| < \|T\|$ , that is,  $\|T^* - \lambda T^{-1}\| < \|T\|$ .  $\square$

The following corollary slightly extends [15, Theorem 1].

**Corollary 3.9.** *Let  $S, T \in \mathcal{B}(\mathcal{H})$  be arbitrary. Then  $0 \in W_T^{\|T\|}(S^*T)$  if and only if  $|\lambda|\|T\| \leq \|S - \lambda T\|$  for all complex numbers  $\lambda$ . In particular, if  $T = V$ , where  $V$  is an isometry, then  $0 \in \overline{W(V^*S)}$  if and only if  $|\lambda| \leq \|S - \lambda V\|$  for all  $\lambda \in \mathbb{C}$ .*

*Proof.* If  $0 \notin W_T^{\|T\|}(S^*T)$ , then  $\text{dist}(T, \mathbb{C}S) < \|T\|$ , by Theorem 3.3. Hence there exists (a non-zero)  $\alpha \in \mathbb{C}$  such that  $\|T - \alpha S\| < \|T\|$ . It follows that  $\|S - \frac{1}{\alpha}T\| < \frac{1}{|\alpha|}\|T\|$ . On the other hand, if there exists  $\lambda \in \mathbb{C}$  such that  $\|S - \lambda T\| < |\lambda|\|T\|$ , then  $\|T - \frac{1}{\lambda}S\| < \|T\|$  which gives  $\text{dist}(T, \mathbb{C}S) < \|T\|$  and therefore, by Theorem 3.3,  $0 \notin W_T(S^*T)$ . For the last assertion observe that  $0 \in \overline{W(V^*S)}$  if and only if  $0 \in \overline{W(S^*V)}$ .  $\square$

**Corollary 3.10.** *If  $T \in \mathcal{B}(\mathcal{H})$  is such that  $0$  is not in the convex hull of the spectrum  $\sigma(T)$ , then there exists an invertible positive operator  $P$  such that  $\text{dist}(T, \mathbb{C}P) < \|T\|$ .*

*Proof.* If  $0$  is not in the convex hull of  $\sigma(T)$ , then, by [4], there exists an invertible positive operator  $P$  such that  $0 \notin \overline{W(PT)}$ . Since  $W_T(PT) \subseteq \overline{W(PT)}$  we conclude that  $\text{dist}(T, \mathbb{C}P) < \|T\|$ .  $\square$

Let  $\mathcal{X}$  be a normed space. Recall that  $x \in \mathcal{X}$  is said to be orthogonal to  $y \in \mathcal{X}$  (in the Birkhoff-James sense) if  $\|x - \lambda y\| \geq \|x\|$  for every scalar  $\lambda$ . If  $x$  is orthogonal to  $y$ , then we write  $x \perp_B y$ . It is well-known that this relation is not symmetric in general.

**Corollary 3.11.** *Let  $P \in \mathcal{B}(\mathcal{H})$  be a positive operator and  $f : \sigma(P) \rightarrow [0, \infty)$  be a continuous function such that  $\|f(P)\| = f(\|P\|)$ . If  $P \perp_B f(P)$ , then  $f(P) \perp_B P$ .*

*Proof.* If  $P \perp_B f(P)$ , then  $\|P - \lambda f(P)\| \geq \|P\|$  for all  $\lambda \in \mathbb{C}$  which means that  $\text{dist}(P, \mathbb{C}f(P)) = \|P\|$ . By Theorem 3.3,  $0 \in W_P^{\|P\|}(f(P)P)$  and therefore, by Proposition 2.7 (iii),  $0 \in W_{f(P)}^{\|P\|}(f(P)P)$ . Since  $P$  and  $f(P)$  commute and  $f(\|P\|) = \|f(P)\|$  we have  $0 \in W_{f(P)}^{\|f(P)\|}(Pf(P))$ . Now we use Theorem 3.3 again and conclude that  $f(P) \perp_B P$ .  $\square$

**Remark 3.12.** A bounded linear operator  $T$  on a normed space  $\mathcal{X}$  is said to have Bhatia-Šemrl property if for any bounded linear operator  $A$  on  $\mathcal{X}$  it follows from  $T \perp_B A$  that there exists a norming vector  $x$  of  $T$ , i.e., a unit vector  $x \in \mathcal{X}$  for which  $\|Tx\| = \|T\|$ , such that  $Tx \perp_B Ax$ . See [13] and reference therein for details about this property. Theorem 3.3 says that on a separable complex Hilbert space  $\mathcal{H}$  every  $T \in \mathcal{B}(\mathcal{H})$  has Bhatia-Šemrl property. Benítez, Fernández and Soriano have proved in [1] that a finite dimensional real normed space is an inner product space if and only if every linear map on it has Bhatia-Šemrl property. It would be interesting to know if their result holds for infinite-dimensional (complex) normed spaces.

By [3, Theorem 1 (ii)], the minimum modulus  $m(T)$  of an operator  $T$  is positive if and only if  $T$  is left invertible, which means that there exists an operator  $L \in \mathcal{B}(\mathcal{H})$  such that  $LT = I$ . In this case,  $m(T) = \frac{1}{\|L\|}$ . If  $T$  is right invertible, then  $T^*$  is left invertible and therefore  $m(T^*) > 0$ . In particular,  $T$  is an invertible operator if and only if  $m(T) > 0$  and  $m(T^*) > 0$ ; in this case  $m(T) = m(T^*) = \frac{1}{\|T^{-1}\|} > 0$ , see [3, Theorem 1].

**Corollary 3.13.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be an invertible operator and  $S \in \mathcal{B}(\mathcal{H})$ . If  $0 \in W_T^{\|T\|}(S^*T)$ , then  $m(T^{-1}) = \sup\{m((T - \lambda S)^{-1}); \lambda \in \mathbb{C} : T - \lambda S \in \mathcal{B}^{-1}\}$ .*

*Proof.* Let  $\lambda \in \mathbb{C}$  be such that  $T - \lambda S$  is invertible. By Theorem 3.3,  $\|T - \lambda S\| \geq \|T\|$  which can be rewritten as  $m((T - \lambda S)^{-1}) \leq m(T^{-1})$ . The assertion follows.  $\square$

We close the paper with a result which gives a characterization of  $\overline{W(S)} \setminus \sigma(S)$  and extends [11, Theorem] from matrices to bounded linear operators.

**Theorem 3.14.** *Let  $S \in \mathcal{B}(\mathcal{H})$ . For  $\lambda \in \mathbb{C} \setminus \sigma(S)$ , the following assertions are equivalent:*

- (i)  $\lambda \in \overline{W(S)} \setminus \sigma(S)$ ;
- (ii)  $\inf_{\mu \in \mathbb{C}} \|I - \mu(S - \lambda I)^{-1}\| = 1$ ;
- (iii)  $\inf_{\mu \in \mathbb{C}} \|(S - \lambda I)^{-1}(S - \mu I)\| = 1$ .

*Proof.* (i) $\Rightarrow$ (ii). Assume that there exists  $\mu \in \mathbb{C}$  such that  $\|I - \mu(S - \lambda I)^{-1}\| < 1$ . Of course,  $\mu \neq 0$ . By Corollary 3.7, there exists  $\alpha \neq 0$  such that  $\|I - \alpha\mu^{-1}(S - \lambda I)\| < 1$ , i.e.,  $\text{dist}(I, \mathbb{C}(S - \lambda I)) < 1$ . It follows, by Corollary 3.4, that  $0 \notin \overline{W(S - \lambda I)}$  and therefore  $\lambda \notin \overline{W(S)}$ .

(ii) $\Rightarrow$ (iii). Assume that  $\|(S - \lambda I)^{-1}(S - \mu I)\| < 1$  for some  $\mu \in \mathbb{C}$ . Then  $\mu \neq \lambda$  and since  $(S - \lambda I)^{-1}(S - \mu I) = I - (\mu - \lambda)(S - \lambda I)^{-1}$  one has  $\|I - (\mu - \lambda)(S - \lambda I)^{-1}\| < 1$ .

(iii) $\Rightarrow$ (i). Suppose that  $\lambda \notin \overline{W(S)}$ . Then  $0 \notin \overline{W(S - \lambda I)}$  and, by Corollary 3.9, there exists  $\beta \in \mathbb{C}$  such that  $|\beta| > \|(S - \lambda I) - \beta I\|$ . This gives  $\|I - \frac{1}{\beta}(S - \lambda I)\| < 1$ . By Corollary 3.7, there exists  $\alpha \in \mathbb{C}$  such that  $\|I - \alpha\beta(S - \lambda I)^{-1}\| < 1$ . It follows, for  $\mu = \lambda + \alpha\beta$ , that  $\|(S - \lambda I)^{-1}(S - \mu I)\| = \|I - \alpha\beta(S - \lambda I)^{-1}\| < 1$ .  $\square$



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