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# Toeplitz operators of finite interval type and the table method

M.C. Câmara<sup>a</sup>, C. Diogo<sup>a,b</sup>, I. M. Spitkovsky<sup>c,d,\*</sup>

<sup>a</sup>*Center for Mathematical Analysis, Geometry, and Dynamical Systems  
Mathematics Department, Instituto Superior Técnico, Universidade de Lisboa  
Av. Rovisco Pais 1049-001 Lisboa, Portugal*

<sup>b</sup>*Departamento de Matemática, Instituto Universitário de Lisboa,  
Av. das Forças Armadas 1649-026 Lisboa, Portugal*

<sup>c</sup>*Division of Science and Mathematics, New York University Abu Dhabi, UAE*

<sup>d</sup>*Department of Mathematics, College of William and Mary,  
Williamsburg, VA 23187, USA*

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## Abstract

We solve a Riemann-Hilbert problem with almost periodic coefficient  $G$ , associated to a Toeplitz operator  $T_G$  in a class which is closely connected to finite interval convolution equations, based on a generalization of the so-called table method. The explicit determination of solutions to that problem allows one to establish necessary and sufficient conditions for the invertibility of the corresponding Toeplitz operator, and to determine an appropriate factorization of  $G$ , providing explicit formulas for the inverse of  $T_G$ . Some unexpected properties of the Fourier spectrum of the solutions are revealed which are not apparent through other approaches to the same problem.

*Keywords:* Toeplitz operator, Riemann-Hilbert problem, Factorization theory, Almost periodic function.

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## 1. Introduction

For  $p > 0$ , let  $H_p^\pm = H^p(\mathbb{C}^\pm)$  denote the Hardy spaces of the upper/lower half-planes  $\mathbb{C}^\pm$ , and let  $L_p := L_p(\mathbb{R})$ . Let moreover  $e_\lambda$  be the function defined by  $e_\lambda(x) = e^{i\lambda x}$ .

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\*Corresponding author

*Email addresses:* `ccamara@math.ist.utl.pt` (M.C. Câmara),  
`crisrina.diogo@iscte.pt` (C. Diogo), `ims2@nyu.edu`, `ilya@math.wm.edu` (I.  
M. Spitkovsky)

For every class  $X$  of functions introduced so far (or below), let  $X^{m \times n}$  denote the class of  $m \times n$  matrices with entries in  $X$ , and let  $X^m = X^{m \times 1}$ . The diagonal  $n \times n$  matrix with diagonal entries  $f_1, \dots, f_n$  will be denoted by  $\text{diag}[f_1, \dots, f_n]$ .

It is well known that the study of several properties of Toeplitz operators  $T_G : (H_p^+)^n \rightarrow (H_p^+)^n$ , with  $G \in L_\infty^{n \times n}$  and  $1 < p < \infty$ , in particular Fredholmness and invertibility, is closely connected with the study of an associated Riemann-Hilbert problem

$$G\phi_+ = \phi_-, \quad (1.1)$$

where  $\phi_\pm$  belong to certain spaces of analytic functions in  $\mathbb{C}^\pm$ .

In this paper we consider Toeplitz operators with  $2 \times 2$  matrix symbols of the form

$$G = \begin{bmatrix} e^{-\lambda} & 0 \\ g & e_\lambda \end{bmatrix}, \quad g \in L_\infty, \quad \lambda > 0, \quad (1.2)$$

which we call *Toeplitz operators of finite interval type*, given their close connection with convolution operators on a finite interval of length  $\lambda$  (cf. [2]), focusing mainly on the case where the non-diagonal function  $g$  is an almost periodic polynomial, i.e.,  $g \in APP$ .

Recall that *APP* consists, by definition, of all finite linear combinations

$$f = \sum_j c_j e_{\lambda_j} \quad (1.3)$$

with complex  $c_j$  and real  $\lambda_j$ . We will say that the set of all  $\lambda_j$  in (1.3) corresponding to  $c_j \neq 0$  is the *Bohr-Fourier spectrum*  $\text{sp}(f)$  of  $f$ , while the respective coefficients  $c_j$  are its *Bohr-Fourier coefficients*.

For matrix functions of the form (1.2) the problem (1.1) with  $\phi_\pm \in (H_\infty^\pm)^n$  is equivalent to

$$g\phi_{1+} = \phi_{2-} - e_\lambda \phi_{2+} \quad \text{with } \phi_{1+}, \phi_{2+} \in H_\infty^+, \quad e_{-\lambda} \phi_{1+}, \phi_{2-} \in H_\infty^-. \quad (1.4)$$

It is clear that, if a function  $\phi_{1+}$  satisfying (1.4) exists, then it determines  $\phi_{1-}$  and  $\phi_{2\pm}$  uniquely. Analogously, if  $\phi_{1-}$  exists, then it determines  $\phi_{1+}$  and  $\phi_{2\pm}$  uniquely. Since  $\phi_\pm$  are completely defined by either  $\phi_{1+}$  or  $\phi_{1-}$ , we will say that  $\phi_{1+}$  (or  $\phi_{1-}$ ) is a solution to the Riemann-Hilbert problem (1.4).

One of the main goals of this paper is to obtain, whenever possible, explicit solutions to (1.4) (or, equivalently, (1.1)) for almost periodic polynomials  $g$  satisfying  $\text{sp}(g) \subset \alpha\mathbb{Z} + \beta\mathbb{Z}$ , with particular emphasis on the case where  $g$  is a trinomial of the form

$$g = ce_{-\sigma} + be_\mu + ae_\alpha, \quad -\sigma < \mu < \alpha, \quad abc \neq 0. \quad (1.5)$$

Our approach to this problem is based on the so-called *table method* which was first presented in [5] and was later extended and developed in [8], as a systematic procedure to obtain explicit solutions of (1.4) with

$$g = c_0 e_{-\beta} + b + \sum_{j=1}^n a_j e_{j\alpha} \quad \text{or} \quad g = \sum_{j=1}^n c_j e_{-j\beta} + b + a_0 e_{\alpha}, \quad (1.6)$$

$0 < \alpha, \beta < \lambda$  and  $b, a_j, c_j \in \mathbb{C}$  ( $j = 0, 1, \dots, n$ ).

It allowed to construct solutions that were completely explicit and, moreover, involved almost periodic functions with what might be regarded as a minimal Bohr-Fourier spectrum.

Since this method is based on a graphical algorithm using a two-entries table, an essential condition for the table method to be applicable is the existence of solutions with spectra supported in an additive subgroup  $x\mathbb{Z} + y\mathbb{Z}$  of  $\mathbb{R}$  with two generators  $x$  and  $y$  ( $\alpha$  and  $\beta$  in the cases studied in [5, 8]), so that the values of the integer coefficients of the (real) parameters  $x$  and  $y$  can be represented in the two entries of the table.

Although it was clear in [5, 8] that the table method was not exhausted by the classes of problems treated in those papers, there was no hint at that point that it could also be used to study Riemann-Hilbert problems of the type (1.4) with  $0 \notin \text{sp}(g)$ . This prompted the question, raised in [5], of characterizing the most general class of *APP* functions  $g$  with spectrum in  $\alpha\mathbb{Z} + \beta\mathbb{Z}$  such that the problem (1.4), with  $g$  given by (1.6), admits an almost periodic solution with spectrum also in  $\alpha\mathbb{Z} + \beta\mathbb{Z}$ .

In fact, it is not difficult to see that the table method, as presented in [5, 8], cannot be applied if  $g$  is given by (1.5) with say  $\alpha + \sigma > \lambda$ ,  $\mu > 0$ . In this paper we show however that problems of the form (1.4) with  $0 \notin \text{sp}(g)$  can be tackled by retaining the essential reasoning underlying the table method, while changing some of its aspects whose importance actually stemmed from the specific properties of the examples studied in the past. It should be stressed, however, that the latter aspects, and the appropriate changes, were by no means evident from the previous works, and overcoming this difficulty was not a trivial task.

By using the table method approach, the solutions thus obtained exhibit certain unexpected properties regarding their Bohr-Fourier spectrum. This allows to consider them optimal, in the sense that they are defined by a function with spectrum in a two parameter additive group. This is all the more surprising given that the spectrum of the elements in  $G$  depend on four parameters and, in particular,  $\text{sp } g$  lies in the three-parameter group  $\alpha\mathbb{Z} + \mu\mathbb{Z} + \sigma\mathbb{Z}$ .

These results are presented in Sections 3 and 4, central to our paper. Namely, in Section 3 we review the essentials of the table method and discuss its implementation in the context of this paper. This (non trivial) generalization of the table method is illustrated by solving a scalar problem (1.4), called Problem  $g$ , for a particular case with trinomial  $g$ . The explicit determination of solutions to Problem  $g$ , for  $g$  given by (1.5), under certain additional restrictions, is obtained in Section 4.

The reason for imposing these restrictions is explained in Section 2. There we also settle the notation and present the third subject that will play a main role in this paper, along with Toeplitz operators  $T_G$  and the Riemann-Hilbert problem (1.1), namely, the  $(AP)$  factorization of  $G$  and its partial  $(AP)$  indices.

In Section 5 we demonstrate how the results of Section 4 can provide explicit solutions to (1.4) satisfying certain corona type conditions, thus yielding existence criteria for a *canonical* (that is, having zero partial indices) factorization of  $G$  and, under rather general assumptions, formulas of the canonical factorization itself. They also provide expressions for its partial  $AP$  indices if the factorization is not canonical in terms of the parameters  $\alpha, \mu$  and  $\sigma$ . Moreover, new lower estimates of the partial  $AP$  indices are obtained, raising the question whether they hold in a broader context.

These results are used in Section 6 to obtain a complete solution of the factorization problem for  $G$  and the invertibility problem for the respective Toeplitz operator  $T_G$ , for a class of matrix symbols  $G$  with parameters  $\alpha, \mu, \sigma$  in (1.5) lying in a certain domain for which a graphical interpretation of the results is possible.

It is clear from the table method itself that it can also be applied to solve Riemann-Hilbert problems of the form (1.4) where  $\text{sp}(g)$  has more than three points in the same two-parameters group. More importantly, the explicit form of the solutions thus obtained makes it clear that their expressions remain valid for non-constant (and even non almost periodic) coefficients in a certain range, henceforth revealing some stability properties that are yet to be fully understood. These generalizations, and related open problems are presented and discussed briefly in the final Section 7.

## 2. Almost periodic symbols and factorization

The algebra  $AP$  of *Bohr almost periodic functions* is defined as the closure of  $APP$ , the set of almost periodic polynomials, with respect to the uniform norm. The notions of Bohr-Fourier spectra and coefficients extend

from  $APP$  to  $AP$ . Namely, the *Bohr-Fourier coefficient*  $\widehat{f}(\lambda)$  is defined as  $\mathbf{M}(e_{-\lambda}f)$ ; recall that the Bohr mean value

$$\mathbf{M}(f) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt$$

exists for any  $f \in AP$ , see e.g. [16, 17] for details. The Bohr-Fourier spectrum  $\text{sp}(f) = \{\lambda: \widehat{f}(\lambda) \neq 0\}$  is at most countable, so the (formal) Bohr-Fourier series  $\sum_{\lambda} \widehat{f}(\lambda)e_{\lambda}$  can be put in correspondence with  $f$ . The set of  $f \in AP$  for which this series converges absolutely, that is,  $\sum_{\lambda} |\widehat{f}(\lambda)| < \infty$ , forms the algebra  $APW$ .

Further, let

$$AP^{\pm} = \{f \in AP: \text{sp}(f) \subset \mathbb{R}_{\pm}\}, \text{ where } \mathbb{R}_{\pm} = \{x \in \mathbb{R}: \pm x \geq 0\}.$$

$AP^{\pm}$  are closed subalgebras of  $AP$ . The subalgebras  $APW^{\pm}$  and  $APP^{\pm}$  are defined as the intersections of  $AP^{\pm}$  with  $APW$  and  $APP$ , respectively. Note that

$$APP^+ \subset APW^+ \subset AP^+ = AP \cap H_{\infty}^+, \quad APP^- \subset APW^- \subset AP^- = AP \cap H_{\infty}^-$$

and so  $AP^+ \cap AP^- = \mathbb{C}$ .

A (right)  $AP$  factorization of an  $n \times n$  matrix function  $G$  is a representation

$$G = G_- D G_+, \tag{2.1}$$

where

$$G_-^{\pm 1} \in (AP^-)^{n \times n}, \quad G_+^{\pm 1} \in (AP^+)^{n \times n},$$

and  $D = \text{diag}[e_{\mu_1}, \dots, e_{\mu_n}]$  with  $\mu_1, \dots, \mu_n \in \mathbb{R}$ . The values  $\mu_1, \dots, \mu_n$  are uniquely defined, up to a permutation, by the factorization (2.1), and are called the *partial AP indices* of  $G$ . Substituting  $AP^{\pm}$  in (2.1) by the more restrictive  $APW^{\pm}$ ,  $APP^{\pm}$ , or the less restrictive  $H_{\infty}^{\pm}$ , we arrive at the definitions of  $APW$ ,  $APP$ , and bounded factorizations of  $G$ , respectively. Either of these factorizations is called *canonical* if in (2.1) all the partial  $AP$  indices are equal to zero, and so the middle factor  $D$  can be dropped:

$$G = G_- G_+. \tag{2.2}$$

If a bounded factorization (2.2) exists, the respective Toeplitz operator is invertible, and (2.2) provides an expression for its inverse:

$$T_G^{-1} = G_+^{-1} P^+ G_-^{-1} I,$$

where  $P^+$  denotes the Riesz projection acting from  $L_p^n$  onto  $(H_p^+)^n$  entry-wise.

For matrix functions  $G \in (APW)^{n \times n}$  this sufficient invertibility condition is also necessary [2, Theorem 5.16]. Moreover, (2.2) is then automatically an *APW* factorization of  $G$ . This is the main reason because of which the *AP* factorization of matrices (1.2) with  $g \in APW$  is of interest. Note that the factorability criterion for such matrices, even with  $g \in APP$ , is presently not known.

Here is a brief summary of what is known for matrix functions of the form (1.2) with  $g$  given by (1.5):

If  $\alpha \geq \lambda$  or  $\sigma \geq \lambda$ , the respective term in (1.5) is inconsequential, and effectively  $g$  becomes, at most, a binomial. If  $\alpha$  or  $\sigma$  are non-positive, then  $\text{sp}(g)$  lies to one side of the origin. Either way,  $G$  is then *APP* factorable, and an explicit factorization was constructed in [12], see also [2], Sections 14.1 and 14.3. Further, if  $(\alpha - \mu)/(\mu + \sigma)$  is rational, then the distances between the points of  $\text{sp}(g)$  are commensurable. This again guarantees the *APP* factorability, with factorization formulas given in [15] and [2, Section 14.4]. We will therefore suppose that

$$0 < \alpha, \sigma < \lambda, \quad \frac{\alpha - \mu}{\mu + \sigma} \notin \mathbb{Q} \quad (2.3)$$

and, without loss of generality, that  $\mu \geq 0$  (see [2, Section 13.2]). We will assume, in addition to (2.3), that

$$\alpha + \sigma \geq \lambda. \quad (2.4)$$

Some factorability results are known for  $\alpha + \sigma < \lambda$ , see e.g. [6], [7], and [8], but we will not pursue this case here.

If in (2.4) the equality holds, i.e., if  $\alpha + \sigma = \lambda$ , and in addition  $\mu = 0$ , then  $G$  is not *APP* factorable. More specifically, it admits a canonical *APW* (but not *APP*) factorization if  $|a|^\sigma |c|^\alpha \neq |b|^\lambda$ , and it is not *AP* factorable otherwise. This criterion was established in [11, 13], while the explicit factorization formulas were obtained in [1]; see also [2], Sections 15.1 and 23.3.

On the other hand, if along with (2.3) we have either  $\alpha + \sigma > \lambda$ , or  $\alpha + \sigma = \lambda$  and  $\mu \neq 0$ , then  $G$  is *APP* factorable. This was shown in [18] (see also [2, Sections 15.2-15.4]) via a recursive procedure, not well suited for the derivation of explicit factorization formulas. The explicit formulas for a canonical factorization of  $G$ , in the case  $\alpha + \sigma > \lambda$ ,  $\mu = 0$  follow as a particular case from [5], where a more general class of almost periodic

polynomials  $g$  was treated. Explicit formulas for the case of a trinomial  $g$  with  $\alpha + \sigma = \lambda$ ,  $\mu \neq 0$  were obtained in [14], showing that this factorization is actually canonical.

Explicit (non-recursive) criteria for existence of a canonical factorization for  $G$ , i.e., invertibility of  $T_G$ , as well as explicit formulas for the factors  $G_{\pm}$ , when

$$\alpha + \sigma > \lambda, \quad \mu \neq 0, \quad (2.5)$$

have not been obtained before. This is why we concentrate in the forthcoming sections on  $g$  given by (1.5) and in addition satisfying (2.3), (2.5).

### 3. The table method and Problem $g$

The factorization problem for  $2 \times 2$  matrices of the form (1.2) is closely connected to the solution of the Riemann-Hilbert problem (1.1) which in its turn, can be equivalently formulated as a scalar problem (1.4). In [5], a Riemann-Hilbert problem of the form (1.4), denoted by Problem  $(A, g)$  where

$$g = ce_{-\sigma} + b + \sum_{j=1}^n a_j e_{j\alpha} \quad \text{or} \quad g = \sum_{j=1}^n c_j e_{-j\sigma} + b + ae_{\alpha}, \quad (3.1)$$

was considered and solved by what might be called a graphical algorithm called the *table method*. Besides its simplicity this method had the advantage of yielding explicit *APP* solutions with coefficients given by rather simple expressions. The reasoning behind the table method, as well as its main steps, have been described in detail in [5, Section 4]. Two main steps were outlined. The first step consisted in obtaining a solution to (1.4) such that  $\phi_{1+} \in APW^+$  with

$$0 \in \text{sp}(\phi_{1+}) \subset \alpha\mathbb{Z} + \sigma\mathbb{Z}, \quad (3.2)$$

this being the starting point. The second step consisted in obtaining a solution with  $\phi_- \in APW^-$  and

$$0 \in \text{sp}(\phi_{1-}) \subset \alpha\mathbb{Z} + \sigma\mathbb{Z}, \quad (3.3)$$

which was linearly independent from the previous one, by applying a simple transformation  $\xi \rightarrow -\xi$  to a solution, satisfying (3.2), of an associate Problem  $(A, g_{(-)})$ , where  $g_{(-)}(\xi) = g(-\xi)$ .

As a consequence it was possible to establish the existence of a canonical factorization of  $G$  in all cases that were considered, as well as the explicit formulas for the factors.



Let now

$$g = ce_{-\sigma} + be_{\mu} + ae_{\alpha} \quad (3.4)$$

with  $a, b, c \in \mathbb{C} \setminus \{0\}$  and

$$\mu \in [0, \lambda[, \quad \sigma, \alpha \in ]0, \lambda[, \quad \alpha > \mu, \quad \alpha + \sigma \geq \lambda, \quad \frac{\alpha - \mu}{\mu + \sigma} \notin \mathbb{Q}, \quad (3.5)$$

assuming moreover that if

$$\alpha + \sigma = \lambda \text{ then } \mu > 0. \quad (3.6)$$

Consider the following:

Problem  $g$ : Determine  $\phi_{1+}, \phi_{2+} \in H_{\infty}^{+}$  and  $\phi_{2-} \in H_{\infty}^{-}$  with  $\text{sp}(\phi_{1+}) \subset [0, \lambda]$ , such that

$$g\phi_{1+} = \phi_{2-} - e_{\lambda}\phi_{2+}$$

for  $g$  satisfying (3.4) and (3.5).

The most obvious difficulty arising in this case is the fact that  $g$  is now a linear combination of exponentials involving three parameters, instead of just two as in the case considered in [5, 8]. On the other hand, as will be shown later, it turns out that in this case it is no longer possible to obtain a solution to the Riemann-Hilbert problem (1.4) satisfying (3.2), nor can we apply a simple change of variables such as  $\xi \rightarrow -\xi$  in order to obtain a second linearly independent solution to the same problem when  $\mu > 0$ . Moreover, as already shown in [2], an  $AP$  factorization of  $G$  in this case is not necessarily canonical.

In order to apply the table method in this case, we start by reducing the Riemann-Hilbert problem (1.4) with  $g$  given by (3.4) to an equivalent problem depending only on two parameters. To this end, let

$$x = \mu + \sigma, \quad y = \alpha - \mu. \quad (3.7)$$

Problem  $g$  can then be restated as either one of the following:

Problem  $(g, r)$ : Determine  $\tilde{\phi}_{1+}, \phi_{2-}, \phi_{2+}$  with  $\tilde{\phi}_{1+}, \phi_{2+} \in APP^{+}$ ,  $\phi_{2-} \in APP^{-}$  and  $\text{sp}(\tilde{\phi}_{1+}) \subset [\alpha - y, \lambda + \alpha - y]$  such that

$$(ce_{-x} + b + ae_y)\tilde{\phi}_{1+} = \phi_{2-} - e_{\lambda}\phi_{2+}.$$

Problem  $(g, v)$ : Determine  $\phi_{1-}, \phi_{2-}, \phi_{2+}$  with  $\phi_{1-}, \phi_{2-} \in APP^{-}$ ,  $\phi_{2+} \in APP^{+}$  and  $\text{sp}(\phi_{1-}) \subset [\lambda, 0]$  such that

$$(ce_{-x} + b + ae_y)\phi_{1-} = e_{-\lambda-\alpha+y}\phi_{2-} - e_{-\alpha+y}\phi_{2+}.$$

Secondly, we replace (3.2) by an equivalent condition which is more appropriate to study the case when  $\mu \neq 0$  in (3.4). Considering for simplicity that  $n = 1$  in (3.1), in which case

$$g = ce_{-\sigma} + b + ae_{\alpha},$$

we easily see that imposing (3.2) is equivalent to imposing that  $\phi_{2-} \in APW^-$  and

$$0 \in \text{sp}(\phi_{2-}) \subset \alpha\mathbb{Z} + \sigma\mathbb{Z}. \quad (3.8)$$

We will show in the next section that it is always possible to find a solution to Problem  $g$  satisfying either (3.3) or (3.8).

Now we present an example which does not involve elaborate computations, in order to illustrate how the results of the following sections were obtained by the table method. Remark however that, while the solutions would have been very difficult to obtain without this graphical algorithm, the proofs of the results in the following sections are all of analytic nature.

Recall that  $(f_{1\pm}, f_{2\pm}) \in (H_{\infty}^{\pm})^2$  is a *corona pair* (cf. [21]) in  $\mathbb{C}^{\pm}$  if and only if

$$\inf_{z \in \mathbb{C}^{\pm}} (|f_{1\pm}(z)| + |f_{2\pm}(z)|) > 0.$$

By the corona theorem (cf. [9]),  $(f_{1\pm}, f_{2\pm})$  satisfies this condition if and only if there exists a pair  $(\tilde{f}_{1\pm}, \tilde{f}_{2\pm}) \in (H_{\infty}^{\pm})^2$  such that  $f_{1\pm}\tilde{f}_{1\pm} + f_{2\pm}\tilde{f}_{2\pm} = 1$  in  $\mathbb{C}^{\pm}$ .

Assume that  $\alpha, \mu, \sigma$  are such that (3.5) holds and, in addition,

$$\frac{3\lambda}{2} \leq \lambda + \alpha \leq 2(\mu + \sigma) \leq \lambda + 2\alpha - \mu. \quad (3.9)$$

In terms of the parameters  $x$  and  $y$  defined by (3.7) we have

$$x + y \geq \lambda, \quad \frac{3\lambda}{2} \leq \lambda + \alpha \leq 2x \leq \lambda + \alpha + y. \quad (3.10)$$

We start by looking for a solution to Problem  $(g, r)$  in the form of a linear combination of exponentials  $e_{jx-ly}$  with  $j, l \in \mathbb{N} \cup \{0\}$ , requiring  $0 \in \text{sp}(\phi_{2-})$ . This implies  $x \in \text{sp}(\tilde{\phi}_{1+})$ . Following the table method, we obtain the results shown in Table 1, where the  $(j, l)$  entry in the boxed area is the Bohr-Fourier coefficient of  $\tilde{\phi}_{1+}$  corresponding to  $e_{jx-ly}$ . The positions marked with \* and \*\* correspond to the points in the spectra of  $\phi_{2-}$  and  $e_{\lambda}\phi_{2+}$  respectively. Note that the point  $2x - 2y$  belongs to both  $\text{sp}(\tilde{\phi}_{1+})$  and  $\text{sp}(e_{\lambda}\phi_{2+})$ .

$j \backslash l$	-1	0	1	2	3
0		*	*	*	
1	**	1	$-\frac{b}{a}$	$\left(-\frac{b}{a}\right)^2$	
2			**	$\left(-\frac{b}{a}\right)^2 \left(-\frac{b}{c}\right)$ **	
3					

Figure 1: Table 1

Thus we have, for Problem  $(g, r)$ ,

$$\tilde{\phi}_{1+} = e_x - \frac{b}{a} e_{x-y} + \frac{b^2}{a^2} e_{x-2y} - \frac{b^3}{a^2 c} e_{2x-2y}, \quad (3.11)$$

which implies that the solution to Problem  $g$  is given by

$$\phi_{1+} = e_{x+y-\alpha} - \frac{b}{a} e_{x-\alpha} + \frac{b^2}{a^2} e_{x-y-\alpha} - \frac{b^3}{a^2 c} e_{2x-y-\alpha}, \quad (3.12)$$

$$\phi_{2+} = -a e_{x+y-\lambda} + \frac{b^3}{ac} e_{2x-y-\lambda} + \frac{b^4}{a^2 c} e_{2x-2y-\lambda}, \quad (3.13)$$

$$\phi_{2-} = c - \frac{bc}{a} e_{-y} + \frac{b^2 c}{a^2} e_{-2y}, \quad (3.14)$$

if

$$x - y \geq \alpha. \quad (3.15)$$

For  $x - y < \alpha$  it is not possible to continue the same procedure and obtain a solution to Problem  $(g, r)$  satisfying (3.8). However, we can obtain a solution to Problem  $(g, v)$  for which (3.3) holds, according to the table below. The  $(j, l)$  entry in the boxed area there is the Bohr-Fourier coefficient of  $\phi_{1-}$  corresponding to  $e_{jx-ly}$ , while \* and \*\* correspond to the points in the spectra of  $e_{-\alpha+y}\phi_{2+}$  and  $e_{-\lambda-\alpha+y}\phi_{2-}$  respectively. Note that  $0 \in \text{sp}(\phi_{1-}) \cap \text{sp}(e_{-\alpha+y}\phi_{2+})$ .

$j \setminus l$	0	1	2	3
0	*	1	*	
-1	$-\frac{c}{b}$	$(-\frac{c}{b})(-\frac{a}{b})$	*	
-2	**	**		
-3				

Figure 2: Table 2

Thus we have the following solution of the Problem  $(g, v)$ :

$$\phi_{1-} = 1 - \frac{c}{b}e_{-x} + \frac{ac}{b^2}e_{-x+y}, \quad (3.16)$$

$$\phi_{2-} = -\frac{ac^2}{b^2}e_{-2x+\lambda+\alpha} - \frac{c^2}{b}e_{-2x-y+\lambda+\alpha}, \quad (3.17)$$

$$\phi_{2+} = -be_{\alpha-y} - ae_{\alpha} - \frac{a^2c}{b^2}e_{-x+y+\alpha}. \quad (3.18)$$

For the case when (3.9) and, in addition, (3.15) hold, we see from (3.12)–(3.13) that

$$(\phi_{1+}, \phi_{2+}) = e_{\delta}(\phi_{1+}^c, \phi_{2+}^c),$$

where

$$\delta = \min\{x + y - \lambda, 2x - 2y - \lambda, x - y - \alpha\} \quad (3.19)$$

and  $\phi_{1+}^c, \phi_{2+}^c \in APP^+$ .

On the other hand, as suggested by Table 1, we have

$$\tilde{\phi}_{1+} = \frac{1}{c}e_x\phi_{2-} - \frac{b^3}{a^2c}e_{2x-2y}$$

where  $\tilde{\phi}_{1+} = e_{\alpha-y}\phi_{1+}$ , which shows that

$$\inf_S (|\phi_{1+}^c| + |\phi_{2+}^c|) > 0$$

for any strip of finite width parallel to the real axis (see the proof of Theorem 2.3 in [3]), while

$$\inf_{\mathbb{C}^+ \setminus S} |\phi_{1+}| > 0 \text{ if } \delta = x + y - \alpha, \text{ and}$$

$$\inf_{\mathbb{C}^+ \setminus S} |\phi_{2+}| > 0 \text{ if } \delta = x + y - \lambda \text{ or } \delta = 2x - 2y - \lambda.$$

Therefore  $(\phi_{1+}^c, \phi_{2+}^c)$  is a corona pair in  $\mathbb{C}^+$ , and we can see analogously that  $(\phi_{1-}, \phi_{2-})$  is a corona pair in  $\mathbb{C}^-$ . Consequently,  $G$  admits an *APP* factorization with partial *AP* indices  $\pm\delta$  defined by (3.19) [4, Theorem 3.8]. Similarly, if (3.9) holds and  $x - y \leq \alpha$ , the partial *AP* indices are  $\pm\delta$  with

$$\delta = \min\{\lambda - x, \alpha - y, \alpha + y - x\}.$$

We conclude, in particular, that an *AP* factorization of  $G$ , with  $x, y, \alpha$  satisfying (3.9), is canonical if and only if

$$x + y = \lambda \quad \text{or} \quad x - y = \alpha.$$

Indeed, for these values of  $x, y, \alpha$  we always have  $\alpha > y$ ,  $2x - 2y > \lambda$ , while  $x = \lambda$  if and only if  $x - y = \alpha$ .

Finally, for  $x - y = \alpha$ , (3.12)–(3.14) and (3.16)–(3.18) yield two linearly independent solutions of the Riemann-Hilbert problem (1.1) which define the factors

$$G_{\pm} = [G_{ij}^{\pm}] \tag{3.20}$$

in a canonical factorization (2.2) of  $G$ , where

$$\begin{aligned} G_{11}^+ &= e_{2(x-\alpha)} - \frac{b}{a} e_{x-\alpha} + \frac{b^2}{a^2} - \frac{b^3}{a^2 c} e_x \\ G_{12}^+ &= e_{\lambda} - \frac{c}{b} e_{\lambda-x} + \frac{ac}{b^2} e_{\lambda-\alpha} \\ G_{21}^+ &= -a e_{x+y-\lambda} + \frac{b^3}{ac} e_{2x-y-\lambda} + \frac{b^4}{a^2 c} e_{2x-2y-\lambda} \\ G_{22}^+ &= -b e_{\alpha-y} - a e_{\alpha} - \frac{a^2 c}{b^2} \\ G_{11}^- &= e_{2(x-\alpha)-\lambda} - \frac{b}{a} e_{x-\alpha-\lambda} + \frac{b^2}{a^2} e_{-\lambda} - \frac{b^3}{a^2 c} e_{x-\lambda} \\ G_{12}^- &= 1 - \frac{c}{b} e_{-x} + \frac{ac}{b^2} e_{-\alpha} \\ G_{21}^- &= c - \frac{bc}{a} e_{-y} + \frac{b^2 c}{a^2} e_{-2y} \\ G_{22}^- &= -\frac{ac^2}{b^2} e_{-2x+\lambda+\alpha} - \frac{c^2}{b} e_{-2x-y+\lambda+\alpha}. \end{aligned}$$

#### 4. Subgroup supported solutions to problem $g$ : the trinomial case

Using the notation (3.7) introduced in Section 3, the conditions (3.5)–(3.6) imposed on  $\text{sp}(g)$  can be rewritten as

$$0 < y \leq \alpha < \lambda, \quad \lambda \leq x + y < \lambda + \alpha, \quad x > 0, \quad \frac{x}{y} \notin \mathbb{Q} \quad (4.1)$$

and

$$x + y \neq \lambda \text{ or } y \neq \alpha. \quad (4.2)$$

We define  $\mathcal{P}$  as the set of all triples  $(x, y, \alpha)$  satisfying (4.1) and (4.2).

From  $x + y \geq \lambda$  and  $y \leq \alpha$  it follows that  $x + \alpha \geq \lambda$  and, taking (4.2) into account, we have

$$x + \alpha > \lambda. \quad (4.3)$$

Below we will repeatedly use the standard notation  $[x]$  for the integer part of  $x \in \mathbb{R}$ , that is, the largest integer not exceeding  $x$ . On the other hand,  $\lfloor x \rfloor$  will stand for the largest integer *strictly* smaller than  $x$ :

$$\lfloor x \rfloor = \begin{cases} [x] & \text{if } x \notin \mathbb{Z}, \\ x - 1 & \text{otherwise.} \end{cases}$$

**Lemma 4.1.** *Let  $(x, y, \alpha) \in \mathcal{P}$ . Then for all  $n \in \mathbb{Z}$  we have either*

$$\left\lfloor \frac{\lambda + ny}{x} \right\rfloor = \left\lfloor \frac{\alpha + ny}{x} \right\rfloor,$$

or

$$\left\lfloor \frac{\lambda + ny}{x} \right\rfloor = \left\lfloor \frac{\alpha + ny}{x} \right\rfloor + 1.$$

*Proof.* It is clear that, since  $\alpha \in ]0, \lambda[$ , we have

$$\frac{\alpha + ny}{x} < \frac{\lambda + ny}{x},$$

and so

$$\left\lfloor \frac{\alpha + ny}{x} \right\rfloor \leq \left\lfloor \frac{\lambda + ny}{x} \right\rfloor. \quad (4.4)$$

From (4.3),

$$\frac{\lambda + ny}{x} < \frac{\alpha + ny}{x} + 1. \quad (4.5)$$

Since  $\left\lfloor \frac{\lambda + ny}{x} \right\rfloor$  is an integer, the result follows from (4.4) and (4.5).  $\square$

For any  $(x, y, \alpha) \in \mathcal{P}$ , let

$$J_{1(x,y,\alpha)} := \left\{ j \in \mathbb{N} \cup \{0\} : \left[ \frac{\alpha + jy}{x} \right] = \left[ \frac{\lambda + jy}{x} \right] \quad \text{and} \quad \frac{\lambda + jy}{x} \notin \mathbb{N} \right\}$$

and

$$J_{(x,y,\alpha)} := \left\{ j \in \mathbb{N} \cup \{0\} : \left[ \frac{\alpha + jy}{x} \right] = \left[ \frac{\lambda + jy}{x} \right] \right\}.$$

To prove our next result, the one-dimensional version of Kronecker's theorem will be needed, see e.g. [10]. For convenience of reference, we provide its statement below.

**Theorem 4.2.** *Let  $p$  be a positive irrational number. Then the set  $\{np - [np] : n \in \mathbb{N}\}$  is dense in the interval  $[0, 1]$ .*

**Theorem 4.3.** *For all  $(x, y, \alpha) \in \mathcal{P}$ , we have  $J_{1(x,y,\alpha)} \neq \emptyset$ .*

*Proof.* If  $x + y > \lambda$ , according to Kronecker's Theorem 4.2, there is some  $n \in \mathbb{N} \cup \{0\}$  such that

$$\frac{(n+1)y}{x} - \left[ \frac{(n+1)y}{x} \right] < 1 - \frac{\lambda - y}{x}, \quad (4.6)$$

since  $0 < 1 - \frac{\lambda - y}{x} < 1$ . Thus we have

$$\frac{\lambda + ny}{x} = \frac{(n+1)y}{x} + \frac{\lambda - y}{x} < 1 + \left[ \frac{(n+1)y}{x} \right] \leq 1 + \left[ \frac{\alpha + ny}{x} \right],$$

so  $\left[ \frac{\lambda + ny}{x} \right] < 1 + \left[ \frac{\alpha + ny}{x} \right]$ . Since  $\frac{\lambda + ny}{x} > \frac{\alpha + ny}{x}$ , we cannot have  $\frac{\lambda + ny}{x} \in \mathbb{N}$  and therefore, by Lemma 4.1 we have

$$\left[ \frac{\alpha + ny}{x} \right] = \left[ \frac{\lambda + ny}{x} \right] = \left[ \frac{\lambda + ny}{x} \right].$$

If  $x + y = \lambda$ , from Kronecker's Theorem we have that there is some  $n \in \mathbb{N} \cup \{0\}$  such that

$$\frac{(n+1)y}{x} - \left[ \frac{(n+1)y}{x} \right] > \frac{\lambda - \alpha}{x},$$

where  $0 < \frac{\lambda - \alpha}{x} < 1$ . Since in this case  $\frac{(n+1)y}{x} = \frac{\lambda + ny}{x}$ , we have

$$\frac{\lambda + ny}{x} - \left[ \frac{\lambda + ny}{x} \right] > \frac{\lambda - \alpha}{x}.$$

It follows that  $\left\lfloor \frac{\lambda+ny}{x} \right\rfloor < \frac{\alpha+ny}{x}$ . Therefore, we have  $\left\lfloor \frac{\lambda+ny}{x} \right\rfloor = \left\lfloor \frac{\alpha+ny}{x} \right\rfloor$  and  $\frac{\lambda+ny}{x} \notin \mathbb{N}$ , so  $\left\lfloor \frac{\alpha+ny}{x} \right\rfloor = \left\lfloor \frac{\lambda+ny}{x} \right\rfloor$ .

If  $\frac{\lambda+ny}{x} \in \mathbb{N}$ , then  $\frac{\lambda+ny}{x} - 1 \in \mathbb{N}$  and  $x + y > \lambda$ . From (4.6) we have

$$\frac{\lambda + ny}{x} - 1 < \left\lfloor \frac{(n+1)y}{x} \right\rfloor \leq 1 + \left\lfloor \frac{\alpha + ny}{x} \right\rfloor \leq 1 + \left\lfloor \frac{\lambda + ny}{x} \right\rfloor = 1 + \frac{\lambda + ny}{x}.$$

Therefore we must have

$$\left\lfloor \frac{(n+1)y}{x} \right\rfloor = 1 + \left\lfloor \frac{\alpha + ny}{x} \right\rfloor = 1 + \frac{\lambda + ny}{x}$$

and  $\left\lfloor \frac{\alpha+ny}{x} \right\rfloor = \frac{\lambda+ny}{x}$ , which is impossible because  $\left\lfloor \frac{\alpha+ny}{x} \right\rfloor \leq \frac{\alpha+ny}{x} < \frac{\lambda+ny}{x}$ .  $\square$

Since  $J_{1(x,y,\alpha)} \subset J_{(x,y,\alpha)}$ , we immediately conclude the following:

**Corollary 4.4.** *For all  $(x, y, \alpha) \in \mathcal{P}$ , we have  $J_{(x,y,\alpha)} \neq \emptyset$ .*

Having fixed  $(x, y, \alpha) \in \mathcal{P}$ , let now

$$N_{(x,y,\alpha)} := \min J_{(x,y,\alpha)}, \quad (4.7)$$

$$S_{-1} = 1, \quad (4.8)$$

$$S_l = \left\lfloor \frac{\lambda + ly}{x} \right\rfloor \quad \text{if } l = 0, 1, \dots, N-1, \quad (4.9)$$

$$S_N := \left\lfloor \frac{\alpha + Ny}{x} \right\rfloor = \left\lfloor \frac{\lambda + Ny}{x} \right\rfloor. \quad (4.10)$$

It is clear that

$$\frac{\lambda + Ny}{x} - 1 \leq S_N \leq \frac{\alpha + Ny}{x}, \quad (4.11)$$

and on the other hand we have

$$\frac{\lambda + ly}{x} - 1 \leq \frac{\lambda + ly}{x} + \frac{\alpha - y}{x} - 1 \leq \frac{\alpha + ly}{x} < \frac{\lambda + ly}{x}, \quad (4.12)$$

for all  $l \in \mathbb{N} \cup \{0\}$ . So, the following theorem holds:

**Theorem 4.5.** *For all  $(x, y, \alpha) \in \mathcal{P}$  we have*

$$\frac{\alpha + ly}{x} < S_l < \frac{\lambda + ly}{x} \quad \text{and} \quad S_l = \left\lfloor \frac{\alpha + ly}{x} \right\rfloor + 1 \quad \text{for all } l = 0, \dots, N-1, \quad (4.13)$$

and one of the following must hold:



- (i)  $\frac{\lambda+Ny}{x} - 1 < S_N \leq \frac{\lambda+Ny}{x} + \frac{\alpha-y}{x} - 1,$
- (ii)  $\frac{\lambda+Ny}{x} + \frac{\alpha-y}{x} - 1 \leq S_N < \frac{\alpha+Ny}{x},$
- (iii)  $S_N = \frac{\lambda+Ny}{x} - 1,$
- (iv)  $S_N = \frac{\alpha+Ny}{x}.$

*Proof.* By (4.7), (4.9) and (4.10) we cannot have  $S_l \leq \frac{\alpha+ly}{x}$ , since this would imply that  $\frac{\lambda+ly}{x} \notin \mathbb{N}$ , because otherwise we would have  $S_l = \frac{\lambda+ly}{x} - 1$  and  $\left\lfloor \frac{\lambda+ly}{x} \right\rfloor = \left\lceil \frac{\lambda+ly}{x} \right\rceil = \left\lceil \frac{\alpha+ly}{x} \right\rceil$ , which is impossible for  $l < N$ . On the other hand, we have  $S_l \leq \frac{\lambda+ly}{x}$  but we cannot have  $S_l = \frac{\lambda+ly}{x}$  since this would imply that  $\left\lfloor \frac{\lambda+ly}{x} \right\rfloor = \frac{\lambda+ly}{x} - 1 = \left\lceil \frac{\alpha+ly}{x} \right\rceil$ , which is impossible for  $l < N$ . Therefore (4.13) must hold. The rest follows immediately from (4.11) and (4.12).  $\square$

Remark that we have  $\lambda \leq \lambda + \alpha - y \leq \alpha + x$ .

We can now present a solution to the Riemann-Hilbert problem (1.1). Recall that, by Theorem 4.5, either  $\lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y$  or  $\lambda + \alpha - y \leq (S_N + 1)x - Ny \leq \alpha + x$ .

**Theorem 4.6.** *For all  $(x, y, \alpha) \in \mathcal{P}$ , the Riemann-Hilbert problem (1.1) admits an APP solution  $(\phi_+, \phi_-)$  such that  $0 \in \text{sp}(\phi_{2-})$  or  $0 \in \text{sp}(\phi_{1-})$ . Namely, if*

$$\lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y, \quad (4.14)$$

*then an APP solution to the Riemann-Hilbert problem (1.1) is given by*

$$\phi_{1+}^r = \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l} \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^l e_{jx-(l-1)y-\alpha} + \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_\gamma, \quad (4.15)$$

$$\text{where } \gamma = (S_N + 1)x - (N - 1)y - \alpha, \quad (4.16)$$

$$\begin{aligned} \phi_{2+}^r = & \sum_{l=0}^{N-1} \sum_{j=S_{l+1}}^{S_{l+1}} -a \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^{l+1} e_{jx-ly-\lambda} \\ & -b \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{(S_N+1)x-Ny-\lambda} + \sum_{j=S_{-1}}^{S_0} \frac{ac}{b} \left(-\frac{b}{c}\right)^j e_{jx+y-\lambda} \\ & -a \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{(S_N+1)x-(N-1)y-\lambda}, \end{aligned} \quad (4.17)$$

$$\phi_{1-}^r = e_{-\lambda} \phi_{1+}^r, \quad (4.18)$$

$$\phi_{2-}^r = \sum_{l=0}^N c \left( -\frac{b}{c} \right)^{S_{l-1}-1} \left( -\frac{b}{a} \right)^l e_{(S_{l-1}-1)x-ly}. \quad (4.19)$$

Respectively, if

$$\lambda + \alpha - y \leq (S_N + 1)x - Ny \leq \alpha + x, \quad (4.20)$$

then an APP solution to the Riemann-Hilbert problem (1.1) is given by

$$\phi_{1-}^v = 1 + \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l} \left( -\frac{c}{b} \right)^j \left( -\frac{a}{b} \right)^l e_{-jx+ly}, \quad (4.21)$$

$$\phi_{2-}^v = \sum_{l=0}^N c \left( -\frac{c}{b} \right)^{S_l} \left( -\frac{a}{b} \right)^l e_{\lambda-(S_{l+1})x+(l-1)y+\alpha}, \quad (4.22)$$

$$\phi_{1+}^v = e_{\lambda} \phi_{1-}^v, \quad (4.23)$$

$$\begin{aligned} \phi_{2+}^v = & -be_{\alpha-y} - ae_{\alpha} - a \left( -\frac{c}{b} \right)^{S_N} \left( -\frac{a}{b} \right)^N e_{-S_N x + N y + \alpha} - \\ & - \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l-1} a \left( -\frac{c}{b} \right)^j \left( -\frac{a}{b} \right)^l e_{-jx+ly+\alpha}. \end{aligned} \quad (4.24)$$

Note that in (4.21) we have  $0 \in \text{sp}(\phi_{1-}^v)$ , while  $0 \in \text{sp}(\phi_{2-}^v)$  in (4.19).

To prove Theorem 4.6 we use the following two results.

**Lemma 4.7.** *Let  $(x, y, \alpha) \in \mathcal{P}$ , and let  $N$  and  $S_l$  be defined by (4.9) and (4.10), respectively. Then*

(i)  $0 \leq S_{l-1} \leq S_l$  for all  $l \in \{0, 1, \dots, N\}$ ;

(ii) If  $l \in \{0, 1, \dots, N-1\}$ , then  $\alpha - y < jx - ly < \lambda$ , for all  $j = S_{l-1}, \dots, S_l$ ;

(iii)  $(S_l + 1)x - ly \geq \lambda$ , for all  $l \in \{0, 1, \dots, N-1\}$ ;

(iv)  $(S_{l-1} - 1)x - ly \leq 0$ , for all  $l \in \{0, 1, \dots, N\}$ .

*Proof.* For  $l = 0, 1, \dots, N-1$ , statement (i) follows immediately from (4.9), and for  $l = N$ , if  $\frac{\lambda+Ny}{x} \notin \mathbb{N}$ , from (4.10). On the other hand, if  $\frac{\lambda+Ny}{x} \in \mathbb{N}$ , then  $S_N = \frac{\lambda+Ny}{x} - 1 = \frac{\lambda+Ny-x}{x}$  and

$$S_{N-1} = \left\lceil \frac{\lambda + (N-1)y}{x} \right\rceil = \left\lceil S_N + 1 - \frac{y}{x} \right\rceil \leq S_N + 1 - \frac{y}{x} < S_N + 1.$$

Therefore,  $S_{N-1} \leq S_N$ .

To prove (ii), it suffices to show that

$$S_{l-1}x - ly > \alpha - y, \quad (4.25)$$

$$S_lx - ly < \lambda \quad (4.26)$$

since, for  $j = S_{l-1}, \dots, S_l$ , we have

$$S_{l-1}x - ly \leq jx - ly \leq S_lx - ly.$$

Now, since  $l - 1 < N$ , we have from Theorem 4.5

$$\frac{\alpha + ly}{x} < S_l < \frac{\lambda + ly}{x}, \quad l = 0, \dots, N - 1$$

so that  $S_{l-1} > \frac{\alpha + (l-1)y}{x}$  and  $S_lx - ly < \lambda$ . Thus, (4.25) and (4.26) hold.

In its turn, (iii) easily follows from the definition of  $S_l$ . The same is true for (iv), taking into account that  $\lambda - y \leq x$  because  $x + y \geq \lambda$ .  $\square$

**Theorem 4.8.** *Let  $(x, y, \alpha) \in \mathcal{P}$ , and let  $N$  be defined by (4.7). If*

$$\lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y, \quad (4.27)$$

*then an APP solution to Problem  $(g, r)$  is given by*

$$\begin{aligned} \tilde{\phi}_{1+}^r &= \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l} \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^l e_{jx-ly} \\ &\quad + \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{(S_N+1)x-Ny}, \\ \phi_{2+}^r &= \sum_{l=0}^{N-1} \sum_{j=S_{l+1}}^{S_{l+1}} -a \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^{l+1} e_{jx-ly-\lambda} + \\ &\quad -b \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{(S_N+1)x-Ny-\lambda} + \sum_{j=S_{-1}}^{S_0} \frac{ac}{b} \left(-\frac{b}{c}\right)^j e_{jx+y-\lambda} \\ &\quad -a \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{(S_N+1)x-(N-1)y-\lambda}, \\ \tilde{\phi}_{1-}^r &= e_{-\lambda} \phi_{1+}, \\ \phi_{2-}^r &= \sum_{l=0}^N c \left(-\frac{b}{c}\right)^{S_{l-1}-1} \left(-\frac{b}{a}\right)^l e_{(S_{l-1}-1)x-ly}. \end{aligned}$$

*Proof.* A straightforward computation shows that

$$(ce_{-x} + b + ae_y)\tilde{\phi}_{1+}^r = \phi_{2-}^r - e_\lambda \phi_{2+}^r.$$

To prove that  $\text{sp}(\tilde{\phi}_{1+}^r) \subset [\alpha - y, \lambda + \alpha - y]$ , it suffices to show that for all  $l = 0, \dots, N$  and  $j = S_{l-1}, \dots, S_l$ ,

$$\alpha - y \leq jx - ly \leq \lambda + \alpha - y$$

and

$$\alpha - y \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y. \quad (4.28)$$

If  $l = 0, \dots, N - 1$ , from Lemma 4.7 (ii), we have

$$\alpha - y < jx - ly \leq \lambda \leq \lambda + \alpha - y,$$

for all  $j = S_{l-1}, \dots, S_l$ . If  $l = N$ , since

$$S_{N-1}x - Ny \leq jx - Ny \leq S_Nx - Ny$$

for  $j = S_{l-1}, \dots, S_l$ , it suffices to show that

$$S_{N-1}x - Ny \geq \alpha - y \quad \text{and} \quad S_Nx - Ny \leq \lambda + \alpha - y,$$

which is indeed the case due to Lemma 4.7 and (4.27), respectively. On the other hand, it is easy to see from (4.27), that (4.28) holds.

It remains to prove that  $\phi_{2\pm}^r \in H_\infty^\pm$ . As to  $\phi_{2+}^r$ , we have:

- If  $l = 0, \dots, N - 1$ , we have  $(S_l + 1)x - ly - \lambda \leq jx - ly - \lambda$ , for all  $j = S_l + 1, \dots, S_{l+1}$ . But due to Lemma 4.7,  $(S_l + 1)x - ly - \lambda \geq 0$ .
- From (v) of the same lemma, it follows that  $(S_N + 1)x - Ny - \lambda \geq 0$ .
- Taking into account that  $0 \leq x + y - \lambda \leq jx + y - \lambda$ , for all  $j = S_{-1}, \dots, S_0$  and (4.27), we conclude that  $\phi_{2+}^r \in H_\infty^+$ .

By (iv) of Lemma 4.7, we have  $(S_{l-1} - 1)x - ly \leq 0$ , for all  $l = 0, \dots, N$ . So we conclude that  $\phi_{2-}^r \in H_\infty^-$ .  $\square$

*Proof of Theorem 4.6:* Note that  $\lambda \leq \lambda + \alpha - y \leq \alpha + x$ . So, according to Theorem 4.5 we have either

$$\lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y \quad (4.29)$$

or

$$\lambda + \alpha - y \leq (S_N + 1)x - Ny \leq \alpha + x. \quad (4.30)$$

Let (4.29) hold. The Riemann-Hilbert problem (1.1) can be written in the form

$$\begin{cases} e^{-\lambda}\phi_{1+} = \phi_{1-} \\ (ce_{-x} + b + ae_y)\phi_{1+} = \phi_{2-} - e_\lambda\phi_{2+} \end{cases}$$

where  $\phi_{1+} = e_{\alpha-y}\tilde{\phi}_{1+}$ . Therefore, the Riemann-Hilbert problem (1.1) admits a solution  $\phi_{1+}$  if and only if  $\tilde{\phi}_{1+} = e_{-\alpha+y}\phi_{1+}$  is a solution to Problem  $(g, r)$ . Therefore, the solution to (1.1) immediately follows from the above mentioned equivalence with Problem  $(g, r)$  and from Theorem 4.8; in this case  $0 \in \text{sp}(\phi_{2-}^v)$ .

Let (4.30) hold. We will prove now that

$$\text{sp}(\phi_{1-}^v) \subset [-\lambda, 0] \quad \text{and} \quad \phi_{2\pm}^v \in H_\infty^\pm.$$

- If  $l = 0, \dots, N$ , to prove that  $\text{sp}(\phi_{1-}^v) \subset [-\lambda, 0]$ , it is enough to show that

$$-S_l x + ly \geq -\lambda \tag{4.31}$$

$$-S_{l-1} x + ly \leq 0, \tag{4.32}$$

since, for  $j = S_{l-1}, \dots, S_l$ , we have

$$-S_l x + ly \leq -jx + ly \leq -S_{l-1} x + ly.$$

It is easy to see that (4.31) follows from Lemma 4.7 (v) and (vi).

From (vi) of the same lemma we have  $-S_{l-1}x + (l-1)y < -\alpha$ ,  $l = 0, \dots, N$ . Since  $y \leq \alpha$  we have  $-S_{l-1}x + ly < y - \alpha \leq 0$  and therefore (4.32) holds.

- To prove that  $\phi_{2-}^v \in H_\infty^-$ , we have to show that

$$\lambda - (S_l + 1)x + (l-1)y + \alpha \leq 0, \quad \text{for } l = 0, \dots, N. \tag{4.33}$$

If  $l = 0, \dots, N-1$ , taking into account that  $x + y \geq \lambda$  and Lemma 4.7 (vi), we have

$$S_l x - ly > \alpha \geq \alpha + \lambda - x - y. \tag{4.34}$$

If  $l = N$ , from (4.30) we have also

$$(S_N + 1)x - Ny \geq \lambda + \alpha - y. \tag{4.35}$$

So (4.34) and (4.35) imply (4.33) and we conclude that  $\phi_{2-}^v \in H_\infty^-$ .

- As to  $\phi_{2+}^v$ , due to (4.30) we have that  $-S_N x + Ny + \alpha \geq 0$ , so it is clear that  $\alpha - y, \alpha, -S_N x + Ny + \alpha \geq 0$ . Therefore, it remains to prove that  $-jx + ly - \alpha \geq 0$ , for all  $l = 0, \dots, N$  and  $j = S_{l-1}, \dots, S_l - 1$ . Since  $-jx + ly - \alpha \geq -(S_l - 1)x + ly - \alpha$ , we just have to show that

$$-(S_l - 1)x + ly - \alpha \geq 0, \quad \text{for } l = 0, \dots, N. \quad (4.36)$$

If  $l = 0, \dots, N - 1$ , taking into account that  $x \geq \lambda - \alpha$  and Lemma 4.7 (vi) we have

$$S_l x - ly \leq \lambda \leq \alpha + x. \quad (4.37)$$

If  $l = N$ , from (4.30) we have

$$S_N x - Ny \leq \alpha < \alpha + x. \quad (4.38)$$

Therefore from (4.37) and (4.38) we have (4.36) and we conclude that  $\phi_{2+}^v \in H_\infty^+$ .

Finally, a straightforward computation shows that in fact (4.21)–(4.24) is an *APP* solution to (1.1) with  $0 \in \text{sp}(\phi_{1-}^v)$ .  $\square$

## 5. Partial *AP* indices and canonical factorization

As was already mentioned in Section 2, matrix functions (1.2) with  $g$  defined by (3.4)–(3.6) are *APP* factorable. We will now use Theorem 4.6 to extract some additional information concerning the partial *AP* indices  $\pm\tilde{\delta}$  of this factorization. To this end, observe the following:

- (i) if  $\lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y$ , then:

$$\phi_+ = e_\delta(\phi_{1+}^c, \phi_{2+}^c) \quad \text{with } \phi_{1+}^c, \phi_{2+}^c \in \text{APP}^+,$$

$$\delta = \min(\{\delta_1, \delta_2\} \cup \{S_l x - ly - \alpha, \quad l = 0, \dots, N - 1\}), \quad (5.1)$$

$$\text{with } \delta_1 = (S_N + 1)x - Ny - \lambda, \quad \delta_2 = x + y - \lambda;$$

$$0 \in \text{sp}(\phi_{1+}^c), \quad 0 \in \text{sp}(\phi_{2-}).$$

- (ii) if  $\lambda + \alpha - y \leq (S_N + 1)x - Ny \leq \alpha + x$ , then:

$$\phi_+ = e_\delta(\phi_{1+}^c, \phi_{2+}^c) \quad \text{with } \phi_{1+}^c, \phi_{2+}^c \in \text{APP}^+,$$

$$\delta = \min(\{\delta_1, \delta_2\} \cup \{\lambda - S_l x + ly, \quad l = 0, \dots, N - 1\}), \quad (5.2)$$

$$\text{with } \delta_1 = -S_N x + Ny + \alpha, \quad \delta_2 = \alpha - y;$$

$$0 \in \text{sp}(\phi_{2+}^c), \quad 0 \in \text{sp}(\phi_{1-}).$$

We now conclude:

**Theorem 5.1.** *The partial AP indices  $\pm\delta$  of  $G$  are given by (5.1)–(5.2) if  $(\phi_{1+}^c, \phi_{2+}^c)$  and  $(\phi_{1-}, \phi_{2-})$  are corona pairs. In that case,  $\delta \leq \mu = \alpha - y$ .*

Recall that the Toeplitz operator  $T_G$  with matrix  $n \times n$  symbol  $G$  acts according to the formula

$$T_G f = P_+ G f, \quad f \in X^n.$$

Various settings are possible, depending on the choice of the space  $X$  and the respective meaning of the (acting entry-wise) projection  $P_+$ . In particular,  $X$  may be a Hardy space  $H_p^+$  of functions analytic in the upper half space with  $1 < p < \infty$ ;  $P_+$  is then the projection of  $L_p$  onto  $H_p^+$  parallel to  $H_p^-$ , and  $G$  can be any matrix function in  $L_\infty^{n \times n}$ . For  $G \in AP^{n \times n}$  one may also take  $X$  to be the Besicovitch space, and for  $G \in APW^{n \times n}$  the case  $X = APW^+$  can be considered; see [2] for the detailed treatment. In all the settings mentioned above, the relation between certain properties of  $T_G$  (Fredholmness, one- or two-sided invertibility, dimensions of kernel and cokernel, etc.) and an appropriate factorization of the symbol  $G$  are the same, and therefore we will not specify the spaces in the forthcoming statements.

**Corollary 5.2.** *Let  $(x, y, \alpha) \in \mathcal{P}$  and let  $N$  be defined by (4.7). A necessary condition for  $T_G$  to be invertible is that  $\delta$ , given by (5.1)–(5.2) is equal to 0, i.e.:*

$$\begin{aligned} (i) \quad & (S_N + 1)x - Ny = \lambda, \quad \text{or} \quad x + y = \lambda, \\ & \text{or} \quad S_L x - Ly = \alpha, \quad \text{for some } L \in \{0, \dots, N - 1\}, \\ & \text{if} \\ & \lambda \leq (S_N + 1)x - Ny \leq \lambda + \alpha - y \end{aligned} \tag{5.3}$$

$$\begin{aligned} (ii) \quad & S_N x - Ny = \alpha, \quad \text{or} \quad y = \alpha, \\ & \text{or} \quad S_L x - Ly = \lambda, \quad \text{for some } L \in \{0, \dots, N - 1\}, \\ & \text{if} \\ & \lambda + \alpha - y \leq (S_N + 1)x - Ny \leq \alpha + x. \end{aligned} \tag{5.4}$$

*These conditions are necessary and sufficient if  $(\phi_{1+}, \phi_{2+})$  and  $(\phi_{1-}, \phi_{2-})$  are corona pairs in  $\mathbb{C}^+$  and  $\mathbb{C}^-$ , respectively.*

These results yield some simple criteria for the invertibility of Toeplitz operators  $T_G$ . For example, we have the following, in the notation of the previous theorem.

**Theorem 5.3.** *If  $(x, y, \alpha) \in \mathcal{P}$  and, for some  $n \in \mathbb{N}$ , we have  $x = \frac{\lambda}{n}$  or  $x = \frac{\alpha}{n-1} \geq \frac{\lambda}{n}$ , then  $T_G$  is invertible.*

*Proof.* Let  $x = \frac{\lambda}{n}$  with  $n \in \mathbb{N}$ . Then  $n - 1 \leq \frac{\alpha}{x} < n$  and we have  $N = 0$ , since  $\lfloor \frac{\lambda}{x} \rfloor = n - 1 = \lfloor \frac{\alpha}{x} \rfloor$ ,  $S_0 = n - 1$ , and  $\phi_{2-} = c$ . Therefore  $(\phi_{1-}, \phi_{2-})$  is a corona pair in  $\mathbb{C}^-$ . On the other hand it is clear that (4.14) is satisfied and it follows from Theorem 4.6 and from Theorem 2.3 in [3] that  $(\phi_{1+}^r, \phi_{2+}^r)$ , given by (4.15) and (4.17), is a corona pair in  $\mathbb{C}^+$ .

We can follow a similar reasoning if  $x = \frac{\alpha}{n-1} \geq \frac{\lambda}{n}$ . In this case  $1 < \frac{\lambda}{\alpha} \leq \frac{n}{n-1}$  and  $n - 1 < \frac{\lambda}{x} = \frac{\lambda}{\alpha}(n - 1) \leq n$ , so that we also have  $\lfloor \frac{\lambda}{x} \rfloor = n - 1 = \frac{\alpha}{x}$  and  $N = 0$ , and (4.20) is satisfied. □

The existence of a canonical factorization can also be proved by determining two linearly independent solutions to the Riemann-Hilbert problem (1.1) satisfying the conditions of the following theorem. In that case  $\delta$ , given by (5.1)–(5.2), is 0 and  $(\phi_{1\pm}, \phi_{2\pm})$  are corona pairs in  $\mathbb{C}^\pm$ , respectively.

**Theorem 5.4** ([5]). *Let  $\phi_{1+}, \psi_{1+}$  be solutions to Problem (1.1). Then  $G$  admits a canonical bounded factorization (2.2) with*

$$G_- = \begin{bmatrix} \phi_{1-} & \psi_{1-} \\ \phi_{2-} & \psi_{2-} \end{bmatrix}, \quad G_+ = \begin{bmatrix} \phi_{1+} & \psi_{1+} \\ \phi_{2+} & \psi_{2+} \end{bmatrix}^{-1},$$

if, for some sequence  $(\xi_n)$  such that  $\xi_n \in \mathbb{C}^+$  (respectively,  $\mathbb{C}^-$ ) and  $|\xi_n| \rightarrow +\infty$  we have

$$\lim_{n \rightarrow \infty} (\phi_{2+}\psi_{1+} - \psi_{2+}\phi_{1+})(\xi_n) \neq 0,$$

(respectively,  $\lim_{n \rightarrow \infty} (\phi_{2-}\psi_{1-} - \psi_{2-}\phi_{1-})(\xi_n) \neq 0$ ).

Now we have the following.

**Theorem 5.5.** *Under the assumptions of Corollary 5.2, the necessary conditions established in this corollary for  $T_G$  to be invertible are also sufficient if:*

(i) (5.3) holds and

$$S_L x - Ly = \alpha, \text{ for some } L \in \{0, \dots, N - 1\}, \quad (5.5)$$

$$\text{or } (S_N + 1)x - Ny = \lambda, \text{ with } N = 0 \text{ or } \left\lfloor \frac{\lambda - \alpha}{y} \right\rfloor \leq \frac{x - \alpha}{y}, \quad (5.6)$$

$$\text{or } x + y = \lambda.$$



(ii) (5.4) holds and

$$S_L x - Ly = \lambda, \text{ for some } L \in \{0, \dots, N-1\}, \quad (5.7)$$

$$\text{or } S_N x - Ny = \alpha, \text{ with } \left\lfloor \frac{\lambda - \alpha + y}{x} \right\rfloor \leq \frac{\lambda}{x} - 1, \quad (5.8)$$

$$\text{or } y = \alpha.$$

*Proof.* (i) Assume that (5.3) holds. If (5.5) or (5.6) hold, we see from Theorem 5.4 that  $T_G$  is invertible, by taking  $\phi_{1\pm} = \phi_{1\pm}^r$ ,  $\phi_{2\pm} = \phi_{2\pm}^r$ , where  $\phi_{1\pm}^r, \phi_{2\pm}^r$  are defined by (4.15–4.19), and

$$\begin{aligned} \psi_{1-} &= 1 + \sum_{l=0}^L \sum_{j=S_{l-1}}^{S_l} \left(-\frac{c}{b}\right)^j \left(-\frac{a}{b}\right)^l e_{-jx+ly}, \\ \psi_{2-} &= \sum_{l=0}^L c \left(-\frac{c}{b}\right)^{S_l} \left(-\frac{a}{b}\right)^l e_{\lambda-(S_l+1)x+(l-1)y+\alpha}, \\ \psi_{1+} &= e_\lambda \psi_{1-}, \\ \psi_{2+} &= -be_{\alpha-y} - ae_\alpha - a \left(-\frac{c}{b}\right)^{S_L} \left(-\frac{a}{b}\right)^L \\ &\quad - \sum_{l=0}^L \sum_{j=S_{l-1}}^{S_l-1} a \left(-\frac{c}{b}\right)^j \left(-\frac{a}{b}\right)^l e_{-jx+ly+\alpha} \end{aligned}$$

if (5.5) is satisfied, and

$$\begin{aligned} \psi_{1-} &= 1 + \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l} \left(-\frac{c}{b}\right)^j \left(-\frac{a}{b}\right)^l e_{-jx+ly} + \left(-\frac{c}{b}\right)^{S_{N+1}} \left(-\frac{a}{b}\right)^N e_{-\lambda} \\ &\quad + \sum_{j=1}^{k-1} \left(-\frac{c}{b}\right)^{S_{N+1}} \left(-\frac{a}{b}\right)^{N+j} e_{-\lambda+jy}, \\ \psi_{2-} &= \sum_{l=0}^{N-1} c \left(-\frac{c}{b}\right)^{S_l} \left(-\frac{a}{b}\right)^l e_{\lambda-(S_l+1)x+(l-1)y+\alpha} \\ &\quad + c \sum_{j=0}^{k-1} \left(-\frac{c}{b}\right)^{S_{N+1}} \left(-\frac{a}{b}\right)^{N+j} e_{-x+(j-1)y+\alpha}, \\ \psi_{1+} &= e_\lambda \psi_{1-}, \end{aligned}$$

$$\begin{aligned}
\psi_{2+} &= -be_{\alpha-y} - ae_{\alpha} - a \left(-\frac{c}{b}\right)^{S_N} \left(-\frac{a}{b}\right)^N e_{-\lambda+x+\alpha} \\
&\quad - \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l-1} a \left(-\frac{c}{b}\right)^j \left(-\frac{a}{b}\right)^l e_{-jx+ly+\alpha} + \\
&\quad + b \left(-\frac{c}{b}\right)^{S_N+1} \left(-\frac{a}{b}\right)^{N+k} e_{-\lambda+(k-1)y+\alpha}, \\
&\quad \text{where } k = \left\lfloor \frac{\lambda - \alpha}{y} \right\rfloor + 2,
\end{aligned}$$

if (5.6) is satisfied.

Note that  $x + y = \alpha + \sigma$ , so the case  $x + y = \lambda$  is covered by [14].

- (ii) Assume now that (5.4) holds. Then, if (5.7) or (5.8) holds, we conclude from Theorem 5.4 that  $T_G$  is invertible by taking  $\phi_{1\pm} = \phi_{1\pm}^v$ ,  $\phi_{2\pm} = \phi_{2\pm}^v$ , where  $\phi_{1\pm}^v, \phi_{2\pm}^v$  are defined by (4.21)–(4.24), and

$$\begin{aligned}
\psi_{1+} &= \sum_{l=0}^L \sum_{j=S_{l-1}}^{S_l} \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^l e_{jx-(l-1)y-\alpha} + \left(-\frac{b}{c}\right)^{S_L} \left(-\frac{b}{a}\right)^L e_{x+y-\alpha}, \\
\psi_{2+} &= \sum_{l=0}^{L-1} \sum_{j=S_{l+1}}^{S_{l+1}} -c \left(-\frac{b}{c}\right)^j \left(-\frac{b}{a}\right)^l e_{jx-ly-\lambda} - b \left(-\frac{b}{c}\right)^{S_L} \left(-\frac{b}{a}\right)^L \\
&\quad - \sum_{j=S_{-1}}^{S_0} a \left(-\frac{b}{c}\right)^{j-1} e_{jx+y-\lambda} - a \left(-\frac{b}{c}\right)^{S_L} \left(-\frac{b}{a}\right)^L e_y, \\
\psi_{1-} &= e_{-\lambda} \psi_{1+}, \\
\psi_{2-} &= \sum_{l=0}^L c \left(-\frac{b}{c}\right)^{S_{l-1}-1} \left(-\frac{b}{a}\right)^l e_{(S_{l-1}-1)x-ly}.
\end{aligned}$$

if (5.7) is satisfied, and

$$\begin{aligned}
\psi_{1+} &= \sum_{l=0}^N \sum_{j=S_{l-1}}^{S_l} \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^l e_{jx-(l-1)y-\alpha} + \left(-\frac{b}{c}\right)^{S_N-1} \left(-\frac{b}{a}\right)^{N+1} \\
&\quad + \sum_{j=1}^k \left(-\frac{b}{c}\right)^{S_N+j-1} \left(-\frac{b}{a}\right)^{N+1} e_{jx},
\end{aligned}$$

$$\begin{aligned}
\psi_{2+} &= \sum_{l=0}^{N-1} \sum_{j=S_{l+1}}^{S_{l+1}} -a \left(-\frac{b}{c}\right)^{j-1} \left(-\frac{b}{a}\right)^{l+1} e_{jx-ly-\lambda} \\
&\quad - \sum_{j=S_{-1}}^{S_0} a \left(-\frac{b}{c}\right)^{j-1} e_{jx+y-\lambda} - \frac{bc}{a} \left(-\frac{b}{c}\right)^{S_N+k} \left(-\frac{b}{a}\right)^N e_{\alpha-y+kx-\lambda} \\
&\quad - \sum_{j=1}^k a \left(-\frac{b}{c}\right)^{S_N+j-1} \left(-\frac{b}{a}\right)^{N+1} e_{\alpha+jx-\lambda}, \\
\psi_{1-} &= e_{-\lambda} \psi_{1+}, \\
\psi_{2-} &= \frac{c^2}{a} \left(-\frac{b}{c}\right)^{S_N} \left(-\frac{b}{a}\right)^N e_{\alpha-y-x} - \sum_{l=0}^N \frac{c^2}{b} \left(-\frac{b}{c}\right)^{S_{l-1}} \left(-\frac{b}{a}\right)^l e_{(S_{l-1}-1)x-ly},
\end{aligned}$$

if (5.8) is satisfied, where  $k = \left\lfloor \frac{\lambda-\alpha+y}{x} \right\rfloor + 1$ .

If  $y = \alpha$ , that is,  $\mu = 0$ , the operator  $T_G$  is invertible by [5].  $\square$

The proof of Theorem 5.5 provides an explicit canonical factorization for  $G$ , revealing in particular additional information concerning the Bohr-Fourier spectra of the entries of  $G_{\pm}$ . To put this in perspective, recall that according to [19] for any  $G$  admitting an  $AP$  factorization and such that  $\text{sp}(G) \subset \Sigma$  for some additive subgroup of  $\mathbb{R}$ , it is possible to choose a factorization in such a way that

$$\text{sp}(G_{\pm}), \text{sp}(G_{\pm}^{-1}) \subset \Sigma. \quad (5.9)$$

In particular, the partial  $AP$  indices of  $G$  lie in  $\Sigma$ . If the  $AP$  factorization of  $G$  is a priori canonical, the latter statement is redundant, and the property (5.9) holds for every factorization of  $G$ , as was shown earlier in [1, 20]. For matrix functions (1.2) this observation was strengthened in [6]. Skipping technical details, for which we refer to Theorem 6.1 of [6], the result is as follows: if  $\text{sp}(g) \subset \Sigma_0$  for some subgroup  $\Sigma_0$  of  $\mathbb{R}$  and (1.2) admits a canonical factorization, then each entry of  $G_{\pm}$  (and thus  $G_{\pm}^{-1}$  as well) has its Bohr-Fourier spectra located in exactly one of the three sets  $\Sigma_0$ ,  $\Sigma_0 + \lambda$  and  $\Sigma_0 - \lambda$ . In our setting of  $g$  given by (1.5),  $\Sigma_0 = x\mathbb{Z} + y\mathbb{Z} + \alpha\mathbb{Z}$ . However, the formulas obtained while proving Theorem 5.5 show that in fact the Bohr-Fourier spectra of each entry of  $G_{\pm}, G_{\pm}^{-1}$  belong to one of five smaller sets  $\Sigma_1, \Sigma_1 \pm \alpha, \Sigma_1 \pm (\alpha + \lambda)$ , where  $\Sigma_1$  is the subgroup  $x\mathbb{Z} + y\mathbb{Z}$  of  $\Sigma_0$ .

**Remark 5.6.** *It remains to be seen whether an  $APP$  factorization of  $G$  is canonical when (5.3) and (5.6) hold without the additional condition that*

$N = 0$  or  $\left\lfloor \frac{\lambda - \alpha}{y} \right\rfloor \leq \frac{x - \alpha}{y}$ , and when (5.4) and (5.8) hold without the additional condition that  $\left\lfloor \frac{\lambda - \alpha + y}{x} \right\rfloor \leq \frac{\lambda}{x} - 1$ . The authors' conjecture is that, in all cases, the necessary conditions of Corollary 5.2 are sufficient for invertibility of  $T_G$ .

## 6. Example

Assume now that  $\alpha, \mu, \sigma$  are such that (3.5) holds, and in addition

$$\alpha > \frac{2}{3}\lambda, \quad \frac{\lambda}{2} \leq x \leq \lambda, \quad (6.1)$$

with  $x$  defined by (3.7). From the results of Section 4, it follows that:

**Theorem 6.1.** *For all  $(x, y, \alpha) \in \mathcal{P}$  satisfying (6.1), the Riemann-Hilbert problem (1.1) admits an APP solution  $(\phi_+, \phi_-)$  given by*

$$\phi_{1+} = \frac{1}{c} e_{x+y-\alpha} (1 - \eta) \phi_{2-} + \zeta, \quad (6.2)$$

$$\phi_{2+} = -e_{-\lambda} P_{[0, +\infty[}(g\phi_{1+}), \quad (6.3)$$

$$\phi_{1-} = e_{-\lambda} \phi_{1+}, \quad (6.4)$$

where

(I) if  $\lambda \leq 2x \leq \alpha + x$ , we have

$$(i) \quad \phi_{2-} = c, \quad \eta = 0, \quad \zeta = -\frac{b}{c} e_{2x+y-\alpha}, \quad \text{if } 2x \leq \lambda + \alpha - y;$$

$$(ii) \quad \phi_{2-} = c \left( -\frac{c}{b} \right) e_{\lambda+\alpha-2x-y}, \quad \eta = 0, \quad \zeta = e_\lambda, \\ \text{if } 2x \geq \lambda + \alpha - y;$$

(II) if  $\alpha \leq x \leq \alpha + y$ , we have

$$(i) \quad \phi_{2-} = c - c \sum_{j=0}^{k-1} \left( \frac{b}{a} \right)^{j+1} \left( \frac{b}{c} \right)^j e_{jx-(j+1)y},$$

$$\eta = \frac{b}{c} e_x, \quad \zeta = \frac{b}{c} e_{2x+y-\alpha},$$

$$\text{if } \lambda \leq (k+1)x - ky \leq \lambda + \alpha - y, \quad k \in \mathbb{N};$$

$$\begin{aligned}
(ii) \quad \phi_{2-} &= c \left(\frac{a}{b}\right)^k \left(\frac{c}{b}\right)^k e_{\lambda+\alpha-(k+1)x+(k-1)y} \\
&\quad - \sum_{j=0}^{k-1} c \left(\frac{a}{b}\right)^j \left(\frac{c}{b}\right)^{j+1} e_{\lambda+\alpha-(j+2)x+(j-1)y}, \\
\eta &= \frac{b}{c} e_x, \quad \zeta = \left(\frac{a}{b}\right)^k \left(\frac{c}{b}\right) e_{\lambda-(k-1)x+ky}, \\
&\text{if } \lambda + \alpha - y \leq (k+1)x - ky \leq \lambda + x - y, \quad k \in \mathbb{N};
\end{aligned}$$

(III) if  $\alpha + y \leq x$ , we have

$$\begin{aligned}
(i) \quad \phi_{2-} &= \sum_{j=0}^{k+1} c \left(-\frac{b}{a}\right)^j e_{-jy}, \\
\eta &= 0, \quad \zeta = \left(-\frac{b}{a}\right)^{k+1} \left(-\frac{b}{c}\right) e_{2x-ky-\alpha}, \\
&\text{if } \alpha + x - y \leq 2x - (k+1)y \leq \lambda + \alpha - y, \quad k \in \mathbb{N};
\end{aligned}$$

$$\begin{aligned}
(ii) \quad \phi_{2-} &= \sum_{j=0}^{k+1} c \left(-\frac{a}{b}\right)^j \left(-\frac{c}{b}\right) e_{\lambda+\alpha-2x+(j-1)y}, \\
\eta &= 0, \quad \zeta = e_\lambda, \\
&\text{if } \lambda + \alpha - y \leq 2x - (k+1)y \leq \alpha + x, \quad k \in \mathbb{N}.
\end{aligned}$$

**Theorem 6.2.** For all  $(x, y, \alpha) \in \mathcal{P}$  satisfying (6.1), the partial AP indices  $\pm\delta$  of  $G$  are given by the following formulas

(I) if  $\lambda \leq 2x \leq \alpha + x$ , we have

$$\begin{aligned}
(i) \quad \delta &= \min\{2x - \lambda, x + y - \lambda\}, \quad \text{if } 2x \leq \lambda + \alpha - y; \\
(ii) \quad \delta &= \min\{\alpha - x, \alpha - y\}, \quad \text{if } 2x \geq \lambda + \alpha - y;
\end{aligned}$$

(II) if  $\alpha \leq x \leq \alpha + y$ , we have

$$\begin{aligned}
(i) \quad \delta &= \min\{(k+1)x - ky - \lambda, x - \alpha\}, \\
&\text{if } \lambda \leq (k+1)x - ky \leq \lambda + \alpha - y, \quad k \in \mathbb{N}; \\
(ii) \quad \delta &= \min\{\lambda - kx + (k-1)y, \alpha - y, \alpha - k(x-y)\}, \\
&\text{if } \lambda + \alpha - y \leq (k+1)x - ky \leq \lambda + x - y, \quad k \in \mathbb{N};
\end{aligned}$$

(III) if  $\alpha + y \leq x$ , we have

- (i)  $\delta = \min\{x - ky - \alpha, x + y - \lambda\}$ ,  
if  $\alpha + x - y \leq 2x - (k + 1)y \leq \lambda + \alpha - y$ ,  $k \in \mathbb{N}$ ;
- (ii)  $\delta = \min\{\lambda - x, \alpha + (k - 1)y - x\}$ ,  
if  $\lambda + \alpha - y \leq 2x - (k + 1)y \leq \alpha + x$ ,  $k \in \mathbb{N}$ .

*Proof.* We prove this result for the case (I)-(i); in the remaining cases the proof is similar, using (6.2) and Theorem 2.3 in [3].

Let  $\lambda \leq 2x \leq \alpha + x$  and  $2x \leq \lambda + \alpha - y$ . Then from (6.2)–(6.4) we obtain

$$\begin{aligned}\phi_{1+} &= e_{x+y-\alpha} + \left(-\frac{b}{c}\right) e_{2x+y-\alpha}, \\ \phi_{2+} &= a \left(-\frac{b}{c}\right) \left(-\frac{b}{a}\right) e_{2x-\lambda} - a e_{x+y-\lambda} - a \left(-\frac{b}{c}\right) e_{2x+y-\lambda}, \\ \phi_{1-} &= e_{-\lambda} \phi_{1+}, \\ \phi_{2-} &= c.\end{aligned}$$

It is easy to check that  $(\phi_{1+}, \phi_{2+}) = e_{\delta}(\phi_{1+}^c, \phi_{2+}^c)$ , where  $\delta = \min\{2x - \lambda, x + y - \lambda\}$  and  $\phi_{1+}^c, \phi_{2+}^c \in APP^+$ . Moreover, by (6.2) we have

$$\inf_S (|\phi_{1+}^c| + |\phi_{2+}^c|) > 0$$

for any strip  $S$  of finite width parallel to the real axis (see the proof of Theorem 2.3 in [3]). In addition,  $\inf_{\mathbb{C}^+ \setminus S} |\phi_{2+}^c| > 0$  if  $\delta = 2x - \lambda$  or  $\delta = x + y - \lambda$ . Therefore  $(\phi_{1+}^c, \phi_{2+}^c)$  is a corona pair in  $\mathbb{C}^+$ . We can see analogously that  $(\phi_{1-}, \phi_{2-})$  is a corona pair in  $\mathbb{C}^-$ , since  $\inf_S (|\phi_{1-}| + |\phi_{2-}|) > 0$  for any strip  $S$  as above and  $\inf_{\mathbb{C}^- \setminus S} |\phi_{2-}^c| > 0$  (cf. [3, Theorem 2.3]).  $\square$

From here we immediately obtain:

**Corollary 6.3.** *For  $(x, y, \alpha) \in \mathcal{P}$  satisfying (6.1), we have  $0 \leq \delta \leq \mu$ .*

Note that we may have  $\delta = \mu$ , and therefore Corollary 6.3 provides optimal estimate for the partial  $AP$  indices.

**Corollary 6.4.** *Let  $(x, y, \alpha) \in \mathcal{P}$  satisfy (6.1). A necessary and sufficient condition for  $T_G$  to be invertible is that*

(I) *if  $\lambda \leq 2x \leq \alpha + x$ , we have*

- (i)  $x = \frac{\lambda}{2}$  or  $x + y = \lambda$ , if  $2x \leq \lambda + \alpha - y$ ;
- (ii)  $x = \alpha$ , if  $2x \geq \lambda + \alpha - y$ ;

(II) if  $\alpha \leq x \leq \alpha + y$ , we have

- (i)  $(k+1)x - ky = \lambda$  or  $x = \alpha$ ,  
if  $\lambda \leq (k+1)x - ky \leq \lambda + \alpha - y$ ,  $k \in \mathbb{N}$ ;
- (ii)  $kx - (k-1)y = \lambda$  or  $k(x-y) = \alpha$ ,  
if  $\lambda + \alpha - y \leq (k+1)x - ky \leq \lambda + x - y$ ,  $k \in \mathbb{N}$ ;

(III) if  $\alpha + y \leq x$ , we have

- (i)  $x - ky = \alpha$  or  $x + y = \lambda$ ,  
if  $\alpha + x - y \leq 2x - (k+1)y \leq \lambda + \alpha - y$ ,  $k \in \mathbb{N}$ ;
- (ii)  $x = \lambda$  or  $x - (k+1)y = \alpha$ ,  
if  $\lambda + \alpha - y \leq 2x - (k+1)y \leq \alpha + x$ ,  $k \in \mathbb{N}$ .

The factors in a canonical factorization of  $G$  can be obtained, for  $\frac{\lambda}{2} < x < \lambda$ ,  $x + y > \lambda$ , from Theorem 6.1 noting that, in each case, it provides two linearly independent solutions to Problem  $g$ . Thus, for instance, if  $x - y = \alpha$ , then (6.2)–(6.4) and (II)-(i) provide the first columns of  $G_+$  and  $G_-$ , while (6.2) – (6.4) and (III)-(ii) provide the second columns of these factors (cf. (3.20)).

In Figure 3 we represent the cross-section of  $\mathcal{P}$  satisfying (6.1) for a certain value  $\alpha$  ( $\alpha = 0, 67$ ) indicating by different colours the points corresponding to (I)-(i), (II)-(i), (III)-(i) (red) and to (I)-(ii), (II)-(ii), (III)-(ii) (green). (In the black and white version red and green correspond to dark and light grey, respectively.) The points on the thicker black lines are those corresponding to values of  $(x, y, \alpha)$  for which  $G$  admits a canonical factorization, according to Corollary 6.4. For all other points the results of Theorems 6.1 and 6.2 provide an explicit solution to Problem  $g$  and to the Riemann-Hilbert problem (1.1), as well as explicit formulas for the partial  $AP$  indices.

It may be worth noting that the borderline cases  $\alpha + \sigma = \lambda$  and  $\alpha + \sigma > \lambda$ ,  $\mu = 0$  for which explicit necessary and sufficient conditions for existence of a canonical factorization of  $G$  were previously known, as mentioned in Section 1, correspond only to the boundary lines of the polygon which are given by the equations  $x + y = \lambda$ ,  $y = \alpha$  in the  $(x, y)$  plane.

## 7. Final remarks

### 7.1. More general $AP$ polynomials

The table method approach is by no means exhausted by the class of symbols studied in the previous sections. The following examples illustrate this point.

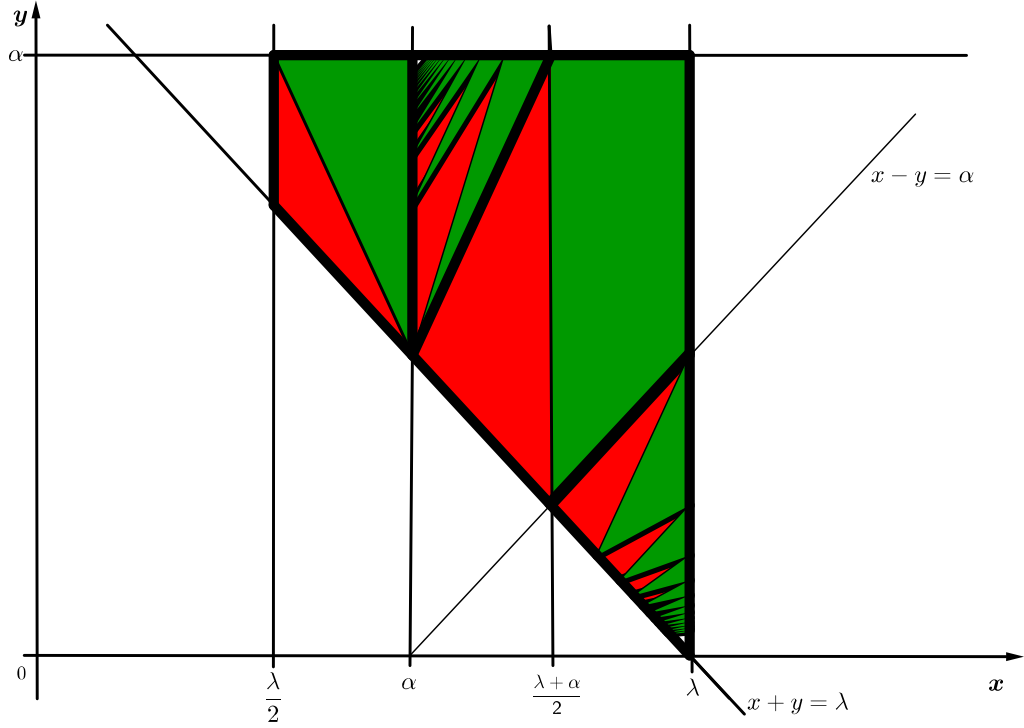


Figure 3: Cross-section for  $\alpha = 0,67$ .

**Example 7.1** Assume that

$$g = ce_{-x} + b + ae_y + de_{2y}$$

with  $a, b, c, d \in \mathbb{C} \setminus \{0\}$ ,  $x, y, \alpha$  for which (3.10) holds and  $\alpha > \frac{2}{3}\lambda$ .

Analogously to what was done in Tables 1 and 2 in Section 3, we see that a solution to Problem  $g$ , and to the Riemann-Hilbert problem (1.1), is given by

$$\begin{aligned} \phi_{1+} = & e_{x+y-\alpha} - \frac{ab}{bd-a^2}e_{x-\alpha} - \frac{b^2}{bd-a^2}e_{x-y-\alpha} \\ & - \frac{b^3}{bdc-a^2c}e_{2x-y-\alpha}, \end{aligned} \quad (7.1)$$



$$\begin{aligned}\phi_{2+} &= \left(-a - \frac{abd}{bd - a^2}\right) e_{x+y-\lambda} - \frac{ab^3}{bdc - a^2} e_{2x-y-\lambda} \\ &\quad + \frac{b^4}{bdc - a^2c} e_{2x-2y-\lambda} - de_{x+2y-\lambda} - \frac{db^3}{bcd - a^2c} e_{2x-\lambda},\end{aligned}\quad (7.2)$$

$$\phi_{2-} = c + \frac{abc}{bd - a^2} e_{-y} - \frac{b^2c}{bd - a^2} e_{-2y},\quad (7.3)$$

if  $x - y \geq \alpha$ , and

$$\phi_{1-} = 1 - \frac{c}{b} e_{-x} + \frac{ac}{b^2} e_{-x+y},\quad (7.4)$$

$$\phi_{2-} = -\frac{ac^2}{b^2} e_{-2x+\lambda+\alpha} - \frac{c^2}{b} e_{-2x-y+\lambda+\alpha},\quad (7.5)$$

$$\begin{aligned}\phi_{2+} &= -be_{\alpha-y} - ae_{\alpha} - \frac{a^2c - bcd}{b^2} e_{-x+y+\alpha} \\ &\quad - de_{y+\alpha} - \frac{acd}{b^2} e_{-x+2y+\alpha},\end{aligned}\quad (7.6)$$

if  $x - y < \alpha$ .

For  $x - y = \alpha$ , (7.1)–(7.3) and (7.4)–(7.6) yield two linearly independent solutions of  $G\phi_+ = \phi_-$  which define the factors  $G_{\pm}$  in a canonical factorization of  $G$ .

**Example 7.2** Let now

$$g = ce_{-x} + b + ae_y + de_{x-y}$$

with  $a, b, c, d \in \mathbb{C} \setminus \{0\}$ ,  $x, y, \alpha$  satisfying (3.10), and  $\alpha > \frac{2}{3}\lambda$ .

In this case a solution to Problem  $g$  is given by

$$\phi_{1+} = e_{x+y-\alpha} - \frac{b}{a} e_{x-\alpha} + \frac{b^2}{a^2} e_{x-y-\alpha} - \frac{b^3}{a^2c} e_{2x-y-\alpha},\quad (7.7)$$

$$\begin{aligned}\phi_{2+} &= -ae_{x+y-\lambda} - \left(\frac{b^3}{ac} - d\right) e_{2x-y-\lambda} + \frac{b^4}{a^2c} e_{2x-2y-\lambda} \\ &\quad - \frac{bd}{a} e_{2x-2y-\lambda} - \frac{db^2}{a^2} e_{2x-3y-\lambda} + \frac{db^3}{a^2c} e_{3x-3y-\lambda},\end{aligned}\quad (7.8)$$

$$\phi_{2-} = c - \frac{bc}{a} e_{-y} + \frac{b^2c}{a^2} e_{-2y}\quad (7.9)$$

if  $x - y \geq \alpha$ , and

$$\phi_{1-} = 1 - \frac{c}{b}e_{-x} + \frac{ac}{b^2}e_{-x+y}, \quad (7.10)$$

$$\phi_{2-} = -\frac{ac^2}{b^2}e_{-2x+\lambda+\alpha} - \frac{c^2}{b}e_{-2x-y+\lambda+\alpha}, \quad (7.11)$$

$$\begin{aligned} \phi_{2+} = & -\left(b + \frac{dac}{b^2}\right)e_{\alpha-y} - ae_{\alpha} - \frac{a^2c}{b^2}e_{-x+y+\alpha} \\ & -de_{x-2y+\alpha} + \frac{cd}{b}e_{\alpha-2y}, \end{aligned} \quad (7.12)$$

if  $x - y < \alpha$  and  $y < \frac{\alpha}{2}$ .

For  $x - y = \alpha$ , (7.7)–(7.9) and (7.10)–(7.12) yield two linearly independent solutions of  $G\phi_+ = \phi_-$  which define the factors  $G_{\pm}$  in a canonical factorization of  $G$ .

These examples raise several interesting questions such as the following. Can the table method be applied to obtain solutions to the Riemann-Hilbert problem (1.1), with  $G$  given by (1.2), for any APP  $g$ ? Is there always an APP solution to that Riemann-Hilbert problem? What are the optimal solutions with respect to the Bohr-Fourier spectrum, and the best estimate of the partial AP indices in terms of the spectrum of  $g$ ?

### 7.2. Non-AP symbols

Finally, we see that knowing the explicit expressions of the solutions to Problem  $g$  enables one to extend the results to some cases where the constant coefficients of the exponentials in  $g$  are replaced by functions, not even necessarily belonging to AP. To illustrate this point, note for example that in case (I) of Theorem 6.1 a bounded factorization of  $G$  exists, and exactly the same factorization formulas persist, when a constant coefficient  $a$  is replaced by an arbitrary function  $a \in H_{\infty}^+$ . This factorization is in fact AP, APW, or APP if and only if  $a$  belongs respectively to  $AP^+$ ,  $APW^+$ , or  $APP^+$ .

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