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An Alternative Proof for Nash's Bargaining Theorem

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Abstract

This paper offers an alternative proof for the main result in Nash(1950). The proof we use intends, in a short way, to give some insight on how the Nash's axioms bridge to his bargaining solution. In particular, to how they shape the function representing the bargaining choice.

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Over six decades have passed since Nash's (1950) seminal article has been published. Its influence cannot be minimized; it is cited by hundreds of papers and there are numerous important departures and generalizations of the same (see Thomson (2009)). The purpose of this paper is to present an alternative and constructive proof that the Nash axioms characterize the Nash bargaining solution. The proof of this result normally starts by assuming the solution and then confirming it until it coincides with the choice in all sets. This follows Nash (1950)'s presentation, although it is unclear as to how the solution came about. We will arrive at the result without guesses, constructively bringing the initial axioms and the final solution together.

We start by establishing, using the *Pareto optimal* (PO) and *Symmetry* (Sym) axioms, the choice on a particular symmetric set whose Pareto-optimal frontier is given by a line. The *Affine Transformation* (AT) axiom is then used to find the choice on any set with a linear Pareto-optimal frontier, with the obvious conclusion that for a bargaining choice to be well defined in the collection of budget sets, *IIA* is not needed. To further extend the solution to all convex sets, *AT* and *IIA* are used. It will be shown why the bargaining solution relates with the function $f(x_1, x_2) = x_1 x_2$, and, in particular, how the *AT* and *IIA* axioms work together to generalize the result from the collection of budget sets to the whole class of convex sets.

As most definitions and axioms we use are widely employed, no intuition is provided. A vector in $\mathbb{R}^2_+$ will be denoted by a bold letter, usually $\boldsymbol{x}$, and its coordinates by $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x_1, x_2)$. The collection $\mathbb{S}^+$ is the set of the compact and convex subsets $S \subset \mathbb{R}^2_+$ with $S^1 S^2 > 0$, where S^1 is the ideal value for the first player, $S^1 = \max\{x : \exists y \in \mathbb{R}, (x, y) \in S\}$ and S^2 is the ideal value for player 2. A set is *symmetric* if $(x, y) \in S$ implies that $(y, x) \in S$. A set is comprehensive if $\boldsymbol{x} \in S$ implies that $\boldsymbol{y} \in S$ for any $\boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{x}$. The *comprehensive convex hull* of S, $ch(S)$, is the smallest comprehensive and convex set that contains S. For a comprehensive set $S \in \mathbb{S}^+$, the function $y_S : [0, S^1] \to [0, S^2]$ defines the maximum value of y when the first coordinate is x; that is, $(x, y) \in S$ if and only if $0 \leq y \leq y_S(x)$. An affine transformation of a vector $\boldsymbol{x} = (x, y)$ by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is $\boldsymbol{\alpha x} = \big(\alpha_1 x, \alpha_2 y\big)$. An affine transformation of a set S by $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha} S = \big\{\boldsymbol{\alpha x} : \boldsymbol{x} \in S\big\}$.

Throughout this paper, we work, without loss of generality, with normalized bargaining where the threat point is $(0, 0)$. In this case, the bargaining problem is defined for $S \in \mathbb{S}^+$, where S is interpreted as the set of available utilities for the players. The Nash bargaining solution is a function $c : \mathbb{S}^+ \to \mathbb{R}^2_+$ such that each bargaining problem S picks a point $c(S) \in S$, respecting the following axioms: *Pareto Optimality (PO)*, for $S \in \mathbb{S}^+, \nexists \boldsymbol{x} \in S \setminus c(S) : \boldsymbol{x} \geq c(S)$; *Independence of Irrelevant Alternatives (IIA)*, if for $S', S \in \mathbb{S}^+$ with $S' \subseteq S$ and $c(S) \in S'$, then $c(S') = c(S)$; *Symmetry (Sym)*, for symmetric $S \in \mathbb{S}^+$, $c(S)_1 = c(S)_2$; *Affine Transformation (AT)*, for $S \in \mathbb{S}^+$ and $\boldsymbol{\alpha} \in \mathbb{R}^2_+$, the bargaining choice verifies $c(\boldsymbol{\alpha} S) = \boldsymbol{\alpha} c(S)$.

The next lemma can be interpreted as a coherence requirement imposed by *IIA*.

Lemma 1. *$S, S' \in \mathbb{S}^+$, $c(S) \neq c(S')$, and $c(S) \in S'$; then, $c(S') \notin S$.*

Proof. By definition, $c(S) \in S$, and by hypothesis, $c(S) \in S'$; then, $c(S) \in S \cap S' \subseteq S$, and by *IIA*, $c(S \cap S') = c(S)$. Assume, by absurd, that $c(S') \in S$, so that $c(S') \in S \cap S' \subseteq S'$; then, using the *IIA* axiom again, $c(S \cap S') = c(S')$, but given that $c(S) \neq c(S')$, this is a contradiction. $\square$

Theorem 1. *The choice c is such that $c(S) = \arg_{\boldsymbol{x}\in S} \max x_1 x_2$ for every $S \in \mathbb{S}^+$.*

For the proof of this theorem, we start by establishing the choice on a set whose Pareto-optimal frontier is given by a line. The AT *axiom is then used to find the choice on any set with a linear Pareto-optimal frontier. To extend the solution to all convex sets,* AT *and* IIA *are used. To get this extension, we start with any $\boldsymbol{z} \in S$ and carefully pick some points $\boldsymbol{x_k}$ for $k \in \mathbb{N}$, with which we define a piecewise linear and convex function $y_\gamma(\cdot)$. We conclude that if any point $(x, y_\gamma(x)) \in S$, then $\boldsymbol{z}$ is not the bargaining choice at S. We then stretch function y_γ to its limit, finding a new function $y(\cdot)$, and conclude that if any point above the curve $(x, y(x))$ is in S, then $\boldsymbol{z}$ is not the choice $c(S)$. Using this result, we can easily deduce that the choice is $c(S) = \arg_{\boldsymbol{x}\in S} \max x_1 x_2$.*

Proof. Throughout this proof, we assume that any set S is such that $S = ch(S)$. We can assume it without loss of generality because any point in $ch(S) \setminus S$ is Pareto dominated in S; hence, *IIA* and *PO* imply that $c(S) = c\big(ch(S)\big)$.

The set $L \in \mathbb{S}^+$ with frontier y_L, defined by the line that passes through $(2, 0)$ and $(0, 2)$, is symmetric; therefore, by *PO* and *Sym*, $c(L) = (1, 1)$. For any given vector $\boldsymbol{z} = (z_1, z_2) \in \mathbb{R}^2_+$, line $L_0 = \boldsymbol{z}L$ has $c(L_0) = \boldsymbol{z}c(L) = \boldsymbol{z}$ and passes through $(2z_1, 0)$ and $(0, 2z_2)$. Thus, y_{L_0} is described[1] by $y_{L_0}(x) = 2z_2 - (z_2/z_1)x$ for $x \in \big[0, 2z_1\big]$. When $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, we use notation $\boldsymbol{\alpha}^k = (\alpha_1^k, \alpha_2^k)$. The proof is divided into seven claims.

Claim 1. *For any given $\boldsymbol{z} = \boldsymbol{x_0} = c(L_0)$ and $\boldsymbol{\alpha} = \big(\gamma, \frac{\gamma}{2\gamma-1}\big)$, with $1/2 < \gamma < 1$, sets $L_k = \boldsymbol{\alpha}^k L_0$ are such that $c(L_k) = \boldsymbol{x_k} \in L_{k+1}$ for any $k \in \mathbb{N}_0$.*

To prove that $c(L_k) = \boldsymbol{x_k} \in L_{k+1}$, we need the expression for the Pareto-optimal frontier of L_{k+1}. We know that $\boldsymbol{\alpha}^{k+1}(2z_1, 0) = \big(2\gamma^{k+1}z_1, 0\big)$ and $\boldsymbol{\alpha}^{k+1}(0, 2z_2) = \big(0, 2(\frac{\gamma}{2\gamma-1})^{k+1}z_2\big)$ are in L_{k+1}; hence,

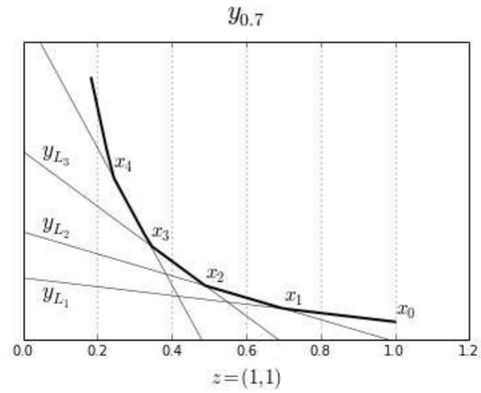
$$y_{L_{k+1}}(x) = \left(\frac{\gamma}{2\gamma-1}\right)^{k+1} 2z_2 - \left(\frac{1}{2\gamma-1}\right)^{k+1} \frac{z_2}{z_1} x \text{ for } x \in [0, 2z_1\gamma^{k+1}].$$

By definition, $\boldsymbol{x_k} = c(L_k) = \boldsymbol{\alpha}^k \boldsymbol{z} = (x_k, y_k) =$, and thus, $x_k = \gamma^k z_1$ and $y_k = (\frac{\gamma}{2\gamma-1})^k z_2$. Then, $y_{L_{k+1}}(\gamma^k z_1) = (\frac{\gamma}{2\gamma-1})^k z_2(2\frac{\gamma}{2\gamma-1} - \frac{1}{2\gamma-1}) = y_k$. Accordingly, $y_{L_{k+1}}(x_k) = y_k$ and $\boldsymbol{x_k} \in L_{k+1}$.

Claim 2. *$\boldsymbol{x_p} \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$, for $1 \leq p < k$, and $p, k \in \mathbb{N}$.*

To prove this claim, we start by defining the piecewise linear curve $y_\gamma(x)$ that passes through all the points $\boldsymbol{x_p}$ for $p \in \mathbb{N}$ and show that this curve is continuous and convex.

[1] Any line that passes through $(a, 0)$ and $(0, b)$ has expression $y(x) = b - (b/a)x$.



For $1/2 < \gamma < 1$ and $0 < x \leq z_1$, if we join the line segments between $\boldsymbol{x_{k-1}} = \boldsymbol{\alpha}^{k-1}\boldsymbol{z}$ and $\boldsymbol{x_k} = \boldsymbol{\alpha}^k\boldsymbol{z}$ in one curve, for all $k \geq 1$, we get curve $y_\gamma(x) = y_{L_k}(x)$ for $\gamma^k z_1 \leq x < \gamma^{k-1}z_1$; that is,

$$y_\gamma(x) = \left(\frac{\gamma}{2\gamma-1}\right)^k 2z_2 - \left(\frac{1}{2\gamma-1}\right)^k \frac{z_2}{z_1}x \text{ for } \gamma^k z_1 \leq x < \gamma^{k-1}z_1 \text{ and } k \in \mathbb{N}. \tag{1}$$

y_γ is continuous: at the interior of each subdomain $\gamma^k z_1 \leq x < \gamma^{k-1}z_1$, y_γ is a line; at the endpoints, as $y_{L_k}(x_{k-1}) = y_{k-1} = y_{L_{k-1}}(x_{k-1})$, the function too is continuous. $y_\gamma(x)$ is convex because its left derivative is non-decreasing (Roberts and Varberg (1973), theorem A, page 10). For $x < x'$, if k and k' are integers such that $\gamma^k z_1 \leq x < \gamma^{k-1}z_1$ and $\gamma^{k'}z_1 \leq x' < \gamma^{k'-1}z_1$, then $\gamma^k z_1 \leq \gamma^{k'}z_1$ and $k \geq k'$, as $1/2 < \gamma < 1$. The left derivative of y_γ at x is equal to $-\left(\frac{1}{2\gamma-1}\right)^k \frac{z_2}{z_1}$, and thus smaller than, the left derivative at x', $-\left(\frac{1}{2\gamma-1}\right)^{k'} \frac{z_2}{z_1}$, because $\frac{1}{2\gamma-1} > 1$, for $1/2 < \gamma < 1$. The left derivative is non-decreasing and $y_\gamma(x)$ is convex.

To show that $\boldsymbol{x_p} = (x_p, y_p) \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$ for $1 \leq p < k$ and $1/2 < \gamma < 1$, note that $x_k = \gamma^k z_1 < \gamma^p z_1 = x_p < x_0$. Then, $x_p = \beta x_0 + (1-\beta)x_k$ for some $\beta \in (0,1)$. By convexity of y_γ, $y_p = y_\gamma(x_p) \leq \beta y_\gamma(x_0) + (1-\beta)y_\gamma(x_k) = \beta y_0 + (1-\beta)y_k = y'_p$. Thus, $\boldsymbol{x_p} = (x_p, y_p) \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$ because $(x_p, y'_p) = \beta\boldsymbol{x_0} + (1-\beta)\boldsymbol{x_k} \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$ and $y_p \leq y'_p$.

Claim 3. $\boldsymbol{x_0} \neq c\big(ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}\big)$ *for* $k \geq 1$.

If $\boldsymbol{x_0} = c\big(ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}\big)$, then $c\big(ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}\big) \in L_1$ by Claim 1. $\boldsymbol{x_1} = c(L_1)$, and by the previous claim, $\boldsymbol{x_1} \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$; therefore, $c(L_1) \in ch\{\boldsymbol{x_0}, \boldsymbol{x_k}\}$. Given that $\boldsymbol{x_0} \neq \boldsymbol{x_1}$, we arrive at a contradiction using Lemma 1.

Claim 4. *For* $x < z_1$, $y(x) = \lim_{\gamma\uparrow 1} y_\gamma(x) = z_1 z_2/x$.

Define $k(\gamma)$ as an integer such that $\gamma^{k(\gamma)}z_1 \leq x < \gamma^{k(\gamma)-1}z_1$. As $\gamma < 1$, $\ln\gamma < 0$. Then, by taking logarithms and manipulating the inequalities, we obtain

$$\ln(x/z_1)/\ln\gamma \leq k(\gamma) < \ln(x/z_1)/\ln\gamma + 1. \tag{2}$$

Knowing that, for $0 < \gamma < 1/2$, $\gamma/(2\gamma-1) > 1$, we have

$$\big(\gamma/(2\gamma-1)\big)^{\ln(x/z_1)/\ln\gamma} \leq \big(\gamma/(2\gamma-1)\big)^{k(\gamma)} \leq \big(\gamma/(2\gamma-1)\big)^{\ln(x/z_1)/\ln\gamma+1}. \tag{3}$$

Taking logarithms and applying L'Hopital's rule, we get the following for the left inequality of (3):

$$\ln(x/z_1)\lim_{\gamma\uparrow 1}\frac{-1/(2\gamma-1)^2}{1/\gamma}=\ln(x/z_1)\lim_{\gamma\uparrow 1}\frac{\ln\big(\gamma/(2\gamma-1)\big)}{\ln\gamma}\leq\lim_{\gamma\uparrow 1}\ln\big(\gamma/(2\gamma-1)\big)^{k(\gamma)}.$$

Doing the same for the right inequality of (3), we conclude that $\lim_{\gamma\uparrow 1}\big(\gamma/(2\gamma-1)\big)^{k(\gamma)}=z_1/x$. Applying the same type of calculations, we get that $\lim_{\gamma\uparrow 1}\big(1/(2\gamma-1)\big)^{k(\gamma)}=(z_1/x)^2$. Substituting these two results into (1), we get that $y(x)=\lim_{\gamma\uparrow 1}y_\gamma(x)=z_1z_2/x$ for $0<x<z_1$.

Claim 5. *For any set $S\in\mathbb{S}^+$ with $\boldsymbol{z}\in S$, if there is an $\boldsymbol{x}=(x_1,x_2)\in S$ such that $0<x_1<z_1$ and $x_1x_2>z_1z_2$, then $c(S)\neq\boldsymbol{z}$.*

If $x_1x_2>z_1z_2$, then, by Claim 4, $x_2>y(x_1)$. Thus, as $(x_1,x_2)\in S$, $0<x_1<z_1$, and $y(x_1)<x_2$, $(x_1,y(x_1))\in\text{int}(S)$. By definition, $k(\gamma)$ is such that $\gamma^{k(\gamma)}z_1\leq x_1<\gamma^{k(\gamma)-1}z_1$. Then, it is easy to conclude that $\lim_{\gamma\to 1}z_1\gamma^{k(\gamma)}=\lim_{\gamma\to 1}\gamma^{k(\gamma)-1}z_1=x_1$. In the previous claim, we proved that

$$\lim_{\gamma\to 1}z_2\big(\gamma/(2\gamma-1)\big)^{k(\gamma)}=z_1z_2/x_1=y(x_1).$$

Hence, $\boldsymbol{x_{k(\gamma)}}=\Big(\gamma^{k(\gamma)}z_1,\big(\gamma/(2\gamma-1)\big)^{k(\gamma)z_2}\Big)=\boldsymbol{\alpha}^{k(\gamma)}\boldsymbol{z}$ converges to $\big(x_1,y(x_1)\big)$ as γ converges to 1. Then, for γ close to 1, $\boldsymbol{x_{k(\gamma)}}=\boldsymbol{\alpha}^{k(\gamma)}\boldsymbol{z}\in\text{int}(S)$. By Claim 3, we know that $\boldsymbol{z}=\boldsymbol{x_0}\neq c(ch\{\boldsymbol{x_0},\boldsymbol{x_{k(\gamma)}}\})$, and given that $\boldsymbol{x_0},\boldsymbol{x_{k(\gamma)}}\in S$, we have $ch\{\boldsymbol{x_0},\boldsymbol{x_{k(\gamma)}}\}\subseteq S$. Then, by *IIA*, $\boldsymbol{z}\neq c(S)$.

Claim 6. *If there is an $\boldsymbol{x}\in S$ such that $x_1>z_1$ and $x_1x_2>z_1z_2$, then $c(S)\neq\boldsymbol{z}$.*

We can apply the same reasoning for $\gamma>1$ and $x>z_1$ that we did for $1/2<\gamma<1$ and $x<z_1$. Defining the piecewise linear function $y_\gamma(x)$ for $x\geq z_1$ and $\gamma>1$ that passes through $\boldsymbol{\alpha}^k\boldsymbol{z}$ for all $k\in\mathbb{N}$, we get

$$y_\gamma(x)=\Big(\frac{\gamma}{2\gamma-1}\Big)^k 2z_2-\Big(\frac{1}{2\gamma-1}\Big)^k\frac{z_2}{z_1}x\text{ for }\gamma^{k-1}z_1\leq x<\gamma^k z_1\text{ and }k\in\mathbb{N}.\tag{4}$$

As before, as $y_\gamma(x)$ is continuous and convex, the conclusion of Claim 3 too holds. Through a similar calculation to that used in Claim 4, it can be derived that $y(x)=\lim_{\gamma\downarrow 1}y_\gamma(x)=z_1z_2/x$. Then, proceeding as in Claim 5, we prove this claim's result.

Claim 7. *For any $S\in\mathbb{S}^+$, $c(S)=\arg_{\boldsymbol{x}\in S}\max x_1x_2$.*

For any $\boldsymbol{z}\in S$, if there is an $\boldsymbol{x}\in S$ such that $x_1x_2>z_1z_2$, then by claims 5 and 6, $\boldsymbol{z}\neq c(S)$. Thus, if it exists, $c(S)$ must be such that $c(S)_1c(S)_2\geq z_1z_2$ for all $\boldsymbol{z}\in S$. Which, given the convexity of $S\in\mathbb{S}^+$, is unique. Thus, the only possible solution is $c(S)=\arg_{\boldsymbol{x}\in S}\max x_1x_2$. It is easy to verify that $\arg_{\boldsymbol{x}\in S}\max x_1x_2$ satisfies all Nash's axioms, hence $c(S)=\arg_{\boldsymbol{x}\in S}\max x_1x_2$ it is in fact the solution. □

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