Economics Bulletin

Volume 30, Issue 3

Diffusion Paths: Fixed Points, Periodicity and Chaos

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Abstract

It is common to recognize that ideas, technology and information disseminate across the economy following some kind of diffusion pattern. Typically, the process of adopting a new piece of knowledge will be translated into an s-shaped trajectory for the adoption rate. This type of process of diffusion tends to be stable in the sense that convergence from any initial state towards the long-term scenario in which all the potential adopters enter in contact with the innovation is commonly guaranteed. Here, we introduce a mechanism under which stability of the diffusion process does not necessarily hold. When the perceived law of motion concerning the evolution of the number of potential adopters differs from the actual law of motion, and agents try to learn this law resorting to an adaptive learning rule, nonlinear long-term outcomes might emerge: the percentage of individuals accepting the innovation in the long-run may be a varying value that evolves according to some cyclical (periodic or a-periodic) pattern. The concept of nonlinear diffusion that is addressed is applied to a problem of information and monetary policy.

Citation: Orlando Gomes, (2010) "Diffusion Paths: Fixed Points, Periodicity and Chaos", *Economics Bulletin*, Vol. 30 no.3 pp. 2413-2424.
Submitted: Aug 07 2010. **Published:** September 16, 2010.

1 Introduction

The establishment of economic and social relations allows ideas, knowledge, technology, information and other intangible entities to spread throughout the interested audiences. At first, because few have yet adopted the innovation, its rate of growth is slow, but it will evolve at an increasing pace; at some stage of this diffusion process, the acceleration phase is typically exhausted and the rate of adoption becomes less intense as the system asymptotically converges to the point representing full adoption. Thus, diffusion is characterized by an s-shaped trajectory for the variable representing the rate of adoption.

Adoption processes have been studied by the economic science under multiple scenarios. Without intending to be exhaustive, we mention some illustrative examples. A first important model of diffusion is the contagion model, used by Bass (1969, 1980) to address the adoption by costumers of new consumption goods; the main idea underlying this model relates to the fact that economic agents tend to adopt an innovation whenever they perceive that others have done the same, i.e., there will be a contagion or an epidemic effect. Following this pioneer work, most of the literature on diffusion concentrates on technology adoption, both in general terms and also relating some specific realities. Relevant generic approaches to the diffusion of technology include, among others, Parente (1994, 2000), Karshenas and Stoneman (1995), Geroski (2000), Chatterjee and Hu (2004) and Mukoyama (2006). An example of technology diffusion in a specific sector is Barros and Martinez-Giralt (2009) who address technological diffusion in health care.

Nevertheless, diffusion processes are not circumscribed to technology adoption. We find, more generally, this type of process associated with social interaction; diffusion of ideas and information inside social networks is a progressively more important theme of discussion by economists. Diffusion in social networks, involving learning or a simple process of influence through direct contact are addressed, e.g., in the work of Bala and Goyal (1998), Young (2003), Cowan and Jonard (2004), Jackson and Yariv (2007) and Richiardi, Gallegati, Greenwald and Stiglitz (2008). Also in macroeconomics, diffusion assumes a relevant role; for instance, Carroll (2006) offers an explanation for infrequent information updating [that is present in the sticky information macro model of Mankiw and Reis (2002, 2006, 2007)] that involves contagion: information spreads across the population as an infectious disease, i.e., under a contagion effect that will imply a typical diffusion process.

The question we ask here is whether convergence to total adoption (or, eventually in some cases, divergence back to a zero adoption level) is the only possible long-term outcome for a typical diffusion process. Assuming a particular bounded rationality setting, we formulate an environment in which agents fail in automatically perceiving the dimension of the interested audience; as a result, they consider a perceived law of motion for the dynamics of the share of potential adopters that eventually differs from the actual law of motion. Estimating the relevant parameter by resorting to an adaptive learning algorithm (involving a constant gain sequence), one may find a nonlinear outcome for the long-term share of agents virtually interested in adopting the innovation. The nonlinearity in potential adoption spills over actual adoption and, in the long-run, we will not be able to find a stable constant share of agents accepting the innovation: this value can change systematically, following either a periodic or an a-periodic pattern of evolution. There is no guarantee that the number of long-term adopters will match the initially predicted quantity of adopters; long-term observed values can be below or above the forecasted level.

If diffusion processes lead to 'boundedly unstable' long-term outcomes, this implies that firms, households or public authorities will experience difficulties in formulating plans or policies. For instance, firms may lack the ability to predict the profitability of a new product or monetary authorities may find it harder to implement policies aimed at stabilizing prices. After characterizing the process through which nonlinear diffusion may occur, we illustrate this process by developing an example involving price setting and the role of monetary policy.

The paper is organized as follows. Section 2 selects a specific type of diffusion process and applies to it the adaptive learning regime. In section 3, we study the underlying dynamics. Section 4 develops the monetary policy example and section 5 concludes.

2 The Diffusion Process

There are several types of dynamic rules that define s-shaped diffusion processes. For instance, Young (2009) makes reference to three sources of diffusion: by contagion, by social influence or by social learning. We will consider the intermediate case - social influence. In this case, each agent will intend to adopt only after a given share of agents have already adopted the innovation.

Let $n_t \in [0,1]$ be the share of agents that at time t have adopted the innovation and consider the cumulative distribution function of this share, $F(n_t)$. The percentage of agents who are willing to adopt but have not done so at time t will be $\widetilde{n}_t F(n_t) - n_t$, with $\widetilde{n}_t \in [0,1]$ the share of potential adopters. Variable $\widetilde{n}_t$ will correspond to the ratio of potential adopters at time t (N_t) relatively to the entire population, which we represent by the constant level $\widetilde{N}$.

Defining a rate of adoption $\lambda \in (0,1)$, the rule governing the diffusion process will be

$$n_{t+1} - n_t = \lambda \left[\widetilde{n}_t F(n_t) - n_t\right], \quad n_0 = 0 \tag{1}$$

Independently of the considered type of distribution, $F : [0,1] \to [0,1]$ and $F' > 0$. Note that F' represents the probability density function of n_t.

We assume that the number of individual agents potentially interested in adopting the new technology or piece of information evolves in time from a given N_0 value towards a steady-state amount $N^* \leq \widetilde{N}$. This evolution process is stable and it will be presented under the form of a simple difference equation,

$$N_{t+1} = aN_t + (1-a)N^*, \quad 0 < a < 1 \tag{2}$$

Dividing both sides of equation (2) by $\widetilde{N}$, we can write the equation under the form

$$\widetilde{n}_{t+1} = a\widetilde{n}_t + (1-a)n^* \tag{3}$$

where $n^* := N^*/\widetilde{N}$. Equation (3) is the actual law of motion (ALM) for the share of interested individuals. Assuming a scenario of perfect foresight, agents know that (3) effectively characterizes the evolution of the number of agents that are potentially attentive to how the adoption rate evolves. Thus, for instance if $\widetilde{n}_0 < n^*$, then $\widetilde{n}_t$ grows positively in time; simultaneously, n_t, i.e., the effective share of adopters, will grow as well. In the long-run, the adoption share will remain at n^*. Note that, in these circumstances, this is a stable outcome. Equation (3) implies convergence towards $\widetilde{n}_t = n^*$, as long as $0 < a < 1$ and, as a consequence, n_t will also converge to the long-run value $n_t = n^*$, i.e., in the steady-state the percentage of agents willing to absorb the ideas / knowledge / news will effectively absorb them.

Now assume that the interested audience ignores the true shape of equation (3). Agents understand that there is an initial share of potentially interested individuals, $\widetilde{n}_0$, but they fail in perceiving that this number converges to n^*. Instead, they will expect a long-term outcome n^{**} that may be distinct from n^*. The perceived law of motion (PLM) is:

$$E_t\widetilde{n}_{t+1} = b\widetilde{n}_t + (1-b)n^{**}, \quad 0 < b < 1 \tag{4}$$

Parameter b represents the expected rate of convergence towards n^{**}, which agents do not know but will try to estimate by observing the time evolution of n_t. The estimation will involve the following learning rule [in Gomes (2009), similar types of learning rules are explored],

$$b_{t+1} = b_t + \sigma(b_{t-1} - b_t), \quad 0 \leq \sigma \leq 1 \tag{5}$$

Equation (5) translates an adaptive learning process under constant gain. Basically, the expression indicates that the extent of the change in the value of b will depend on how is the previous period change weighted. Parameter σ reflects how demanding is the learning process; if $\sigma = 0$, no adjustment is required and b remains constant (this will be the perfect foresight case, where agents possess the sufficient information to take as granted that the actual value of the parameter is the one already considered and no further adjustment is necessary). The higher the value of σ, the larger will be the impact of previous changes in the value of b over the current change.

According to the PLM, $b_t = \frac{E_t\widetilde{n}_{t+1}-n^{**}}{\widetilde{n}_t-n^{**}}$; observe that the value of b in period $t-1$ will be known in t and corresponds to $b_{t-1} = \frac{\widetilde{n}_t-n^{**}}{\widetilde{n}_{t-1}-n^{**}}$. The learning rule can rewritten as

$$\frac{E_{t+1}\widetilde{n}_{t+2} - n^{**}}{\widetilde{n}_{t+1} - n^{**}} = \frac{E_t\widetilde{n}_{t+1} - n^{**}}{\widetilde{n}_t - n^{**}} + \sigma\left(\frac{\widetilde{n}_t - n^{**}}{\widetilde{n}_{t-1} - n^{**}} - \frac{E_t\widetilde{n}_{t+1} - n^{**}}{\widetilde{n}_t - n^{**}}\right) \tag{6}$$

Equation (6) corresponds to how the evolution of $\widetilde{n}_t$ will be perceived by the economic agents. This is the rule they will incorporate in their behavior. However, in period $t+1$, the expectation $E_t\widetilde{n}_{t+1}$ will correspond to the observed value of $\widetilde{n}_{t+1}$ (which is given by the ALM). As a result, the motion of $\widetilde{n}_t$ that agents will consider relevant to take their adoption decisions is (6) with $E_t\widetilde{n}_{t+1} = \widetilde{n}_{t+1}$. The observed motion of $\widetilde{n}_t$ will be given by the pair of equations

$$\begin{cases} \widetilde{n}_{t+1} = \frac{x_t-1}{x_t-a}n^{**} + \frac{1-a}{x_t-a}n^* \\ z_{t+1} = \widetilde{n}_t \end{cases} \tag{7}$$

with $x_t := (1-\sigma)\frac{a\widetilde{n}_t+(1-a)n^*-n^{**}}{\widetilde{n}_t-n^{**}} + \sigma\frac{\widetilde{n}_t-n^{**}}{z_t-n^{**}}$.

3 Local and Global Dynamics

We need to study the dynamic properties of (7) in order to obtain insights about the value that the share of potential adopters will, in fact, display. We begin by addressing local dynamics. The steady-state corresponds to the point $\overline{\widetilde{n}}_t = \overline{z} = n^*$. System (7) can be linearized in the vicinity of this point; we get,

$$\begin{bmatrix} \widetilde{n}_{t+1} - n^* \\ z_{t+1} - n^* \end{bmatrix} = \begin{bmatrix} 1-\sigma-\frac{\sigma}{1-a} & \frac{\sigma}{1-a} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \widetilde{n}_t - n^* \\ z_t - n^* \end{bmatrix} \tag{8}$$

Trace and determinant of the matrix in the linearized system are, respectively, $Tr = 1 - \sigma - \frac{\sigma}{1-a}$ and $Det = -\frac{\sigma}{1-a}$. Stability conditions are straightforward to compute:

(i) $1 - Det = 1 + \frac{\sigma}{1-a} > 0$;

(ii) $1 - Tr + Det = \sigma > 0$;

(iii) $1 + Tr + Det = 2 - \sigma - 2\frac{\sigma}{1-a} > 0$

The first two conditions are satisfied for any admissible value of parameters σ and a. The third condition may not hold. Stability is guaranteed under the inequality $\sigma < \frac{2(1-a)}{3-a}$. At point $\sigma = \frac{2(1-a)}{3-a}$ a flip bifurcation will occur. Therefore, convergence of $\widetilde{n}_t$ to n^* will require a low value for the gain parameter; if a relatively large weight is attributed to previous changes when estimating future values of the parameter in the PLM, the system may be pushed into a region of absence of local stability.

Figure 1 illustrates local dynamics through the presentation of a trace-determinant diagram. The inverted triangle will correspond to the area of stability (it is the area delimited by the three bifurcation lines); the line in bold represents the dynamics of the system (the pairs trace-determinant that are admissible for different values of the gain value). One can observe that for low values of σ stability is evidenced; large values of σ make the system fall outside the stability area.

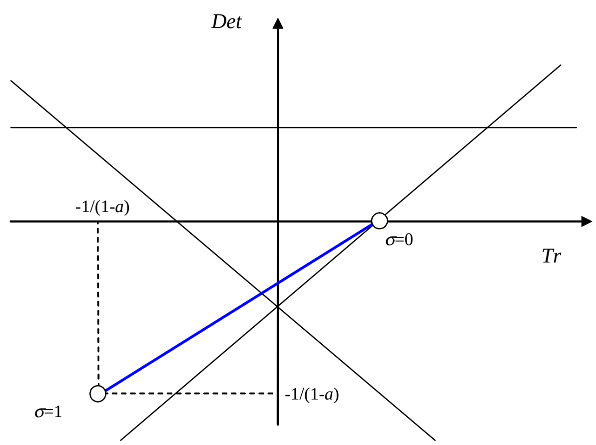

Figure 1. Local Stability in a Trace-Determinant Diagram

Looking at global dynamics, one observes that once stability is lost, cycles of various periodicities are formed for different values of σ. Figure 2 displays a bifurcation diagram drawn for $a = 0.9$. According to the stability condition, stability holds for any value of σ below 0.0952; above this value, we observe the presence of cycles of various periodicities and even the existence of small regions of chaotic motion. Thus, $\widetilde{n}_t$ may not remain at the constant steady-state level (in the figure this is $n^* = 0.9$) and, due to the learning process, the share of potential adopters may register an evolution characterized by the persistence of regular or irregular cycles. In panel **B** of figure 2, a segment of the bifurcation diagram where chaotic motion is identified is displayed in detail.

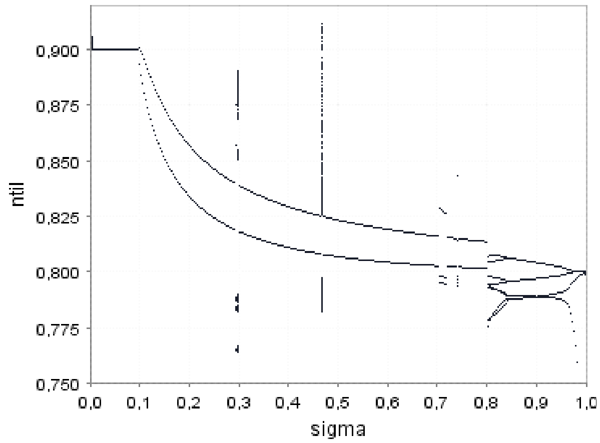

Figure 2. Panel **A.** Bifurcation diagram ($0 < \sigma < 1$).

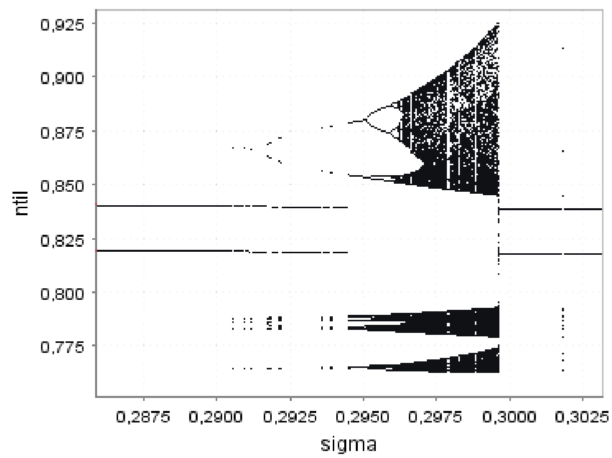

Figure 2. Panel **B**. Bifurcation diagram ($0.2865 < \sigma < 0.303$).

If the target that the diffusion process intends to fulfill is a moving target, then there will not be convergence towards a given constant number of adopters. Figure 3 displays the outcome for n_t when the gain parameter takes the value $\sigma = 0.4$ (and, thus, a period-2 cycle is evidenced); recall that the motion of n_t is given by equation (1). To build the trajectory for n_t, we have considered a normal distribution with mean equal to 0.1 and variance equal to 0.01. Note that because $\widetilde{n}_t$ is subject to endogenous fluctuations, a same type of nonlinear evolution is passed on to the effective adoption share. Two panels are displayed; panel **A** respects to the first 50 observations of n_t; through this panel, we can confirm the s-shaped pattern of the adoption process. Panel **B** displays 50 observations after excluding the transient phase; we observe that the long-term evolution is characterized by a period-2 cycle with n_t assuming, alternatively, the values 0.8186 and 0.8213.

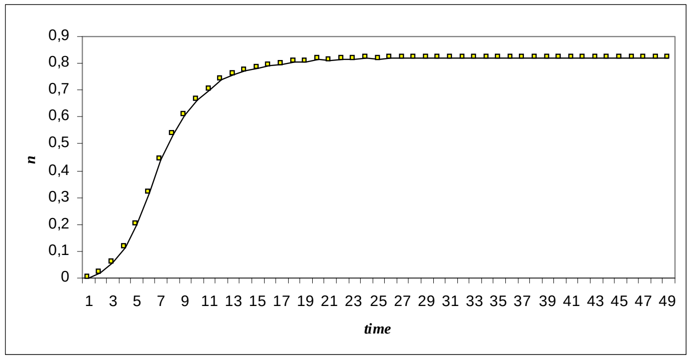

Figure 3. Panel **A**. Diffusion process (first 50 observations).

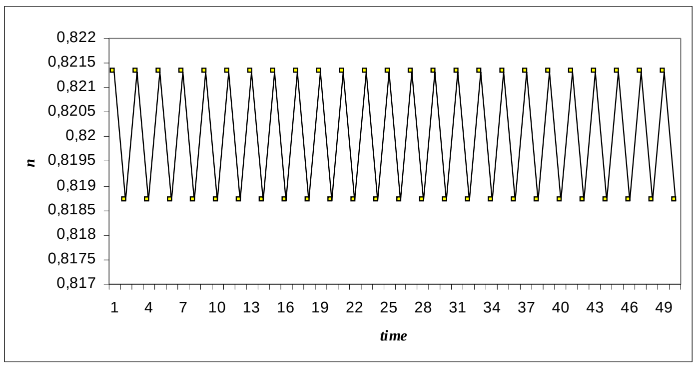

Figure 3. Panel **B**. Diffusion process (long-run behavior).

4 The Monetary Policy Example

To illustrate the nonlinear diffusion result, we apply it to the role of monetary policy in a setting where information about price evolution disseminates gradually across the economy.

Monetary policy will be characterized by the selection of some rate of money growth. The initial policy will be such that money supply grows at the constant rate Δm_0; at a given time moment, the central bank changes its policy by increasing (or decreasing) its rate of growth of money supply to Δm_1. Firms will set prices at time t to the subsequent time period, given their expectations about price evolution.

Two types of expectations are formed: agents who have not yet perceived the change in policy will expect prices to move as they did until the previous time moment, i.e., they form an expectation $E_t^I p_{t+1} = p_t + \Delta m_0$, with p_t the price level at t, measured in logs. The firms that are able to understand how the policy has shifted will form rational expectations, i.e., $E_t^{II} p_{t+1} = p_t^*$, where p_t^* represents the target price firms intend to set under a monopolistically competitive environment [see Mankiw and Reis (2002) for details on the formation of this target price]. The target price is defined by $p_t^* = p_t + \alpha y_t$, where y_t is the output gap (the difference, in logs, between effective and potential output) and $\alpha > 0$ is a parameter representing the degree of real rigidities or the extent in which the different varieties of the assumed good are more or less close from being perfect substitutes (perfect substitutability is represented by $\alpha = 0$). The value of p_t^* is chosen optimally by profit maximizing firms and it indicates how firms react to different stages of the business cycles: phases of expansion push desired prices upward, while phases of recession imply a downward movement of desired prices.

At time t, a share n_t of firms will be acquainted with the policy change while the remainder share $1 - n_t$ will continue to set prices as if the policy had not changed (and therefore re-optimization is not needed). The aggregate price level at time $t + 1$ will be

$$p_{t+1} = (1 - n_t)E_t^I p_{t+1} + n_t E_t^{II} p_{t+1} \tag{9}$$

Given the expectations formation rules, and defining the inflation rate as $\pi_t := p_t - p_{t-1}$, it is straightforward to transform the price level in expression (9) into a Phillips curve (into a relation between the output gap and the inflation rate):

$$\pi_{t+1} = \alpha \frac{n_t}{1 - n_t} y_{t+1} + \Delta m_0 \tag{10}$$

Equation (10) can be interpreted as follows: the larger is the share of agents that, at time t, has already acknowledged the change in policy, the steeper will be the relation, at time $t+1$, between the output gap and the inflation rate.

Next, assume a simple money demand equation $m_t = y_t + p_t$ (demand for money equals nominal output). If the policy change has already occurred, we can apply first differences to this equation to notice that the change in the output gap will correspond to the difference between monetary growth and price change, i.e., $y_{t+1} - y_t = \Delta m_1 - \pi_{t+1}$. Combining this relation with equation (10), we compute a rule of motion for the inflation rate:

$$\pi_{t+1} = \frac{\alpha n_{t-1} n_t \Delta m_1 - (n_t - n_{t-1})\Delta m_0 + (1 - n_{t-1}) n_t \pi_t}{n_{t-1}\left[1 - (1 - \alpha) n_t\right]} \tag{11}$$

Equation (11) reveals the relevant role of information acquisition when assessing the dynamics of the inflation rate. The share of firms that absorb information about monetary policy changes is crucial in determining the path of inflation.

If one assumes that share n_t evolves as characterized in section 3, the eventual nonlinear motion of the adoption rate spills over the long-term inflation trajectory. In this case, the lack of price stability will be the outcome of an unstable information diffusion process that does not allow for the number of firms that accept the information about the new policy as reliable to be a constant value.

Let us first look at the steady-state. Taking $\overline{\pi} := \pi_{t+1} = \pi_t$ and applying this condition to expression (11), we get $\overline{\pi} = \Delta m_1$. Thus, the initial level of inflation will correspond to the rate of money growth before the policy change: $\pi_0 = \Delta m_0$; in the long-run, the inflation rate will coincide with the new money supply growth rate (if the steady-state is a stable point). The convergence from one to the other fixed point will occur gradually, as the diffusion process implies a progressive adoption of the new information by the interested audience. The Phillips curve equation (10) allows us to determine the steady-state for the output gap: $\overline{y} = \frac{1}{\alpha}\frac{1-n^*}{n^*}(\Delta m_1 - \Delta m_0)$; assuming $\Delta m_1 > \Delta m_0$, the steady-state output gap remains positive as long as $n^* < 1$. If all the population accesses the new information, then the output gap falls to zero.

The long-run inflation and output gap trajectories are dependent on the specification of the information diffusion process. If we consider the learning environment, the long-term outcome for the rate of inflation and for the output gap will correspond to a fixed-point, periodic cycles or chaos if each of these is also the outcome of the diffusion process. Deficiencies in efficiently learning the number of agents that are effectively interested in some piece of information, may lead to nonlinear outcomes for both inflation and real economic activity measures. Figure 4 illustrates the result for the inflation rate. The example is the same used to draw figure 3, i.e., we consider $\sigma = 0.4$ and, therefore, the long-term outcome is characterized by a period-2 cycle (in this case, the inflation rate switches between 0.02992 and 0.03007); in this example, it is also considered that $\Delta m_0 = 0.02$ and $\Delta m_1 = 0.03$. Panel **A** shows the transient phase, where the information diffusion process implies an inflation 'overshooting'; the long-run result (shown in panel **B**) reveals that, in fact, a period-2 cycle is followed. It is relevant to observe that the inflation rate fluctuates around the long-term growth rate of money supply, which is 0.03.

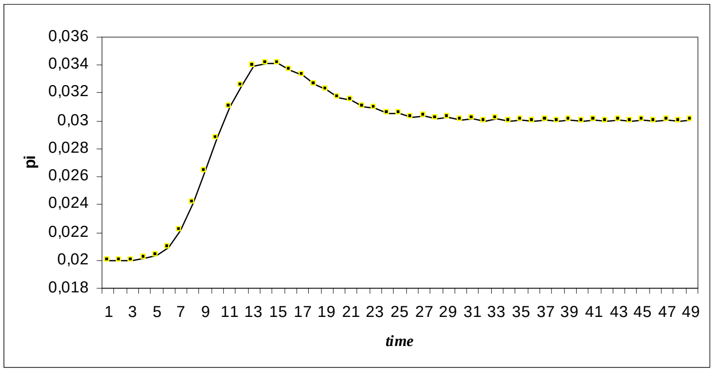

Figure 4. Panel **A**. The effect of a monetary policy change over the inflation rate (first 50 observations; $\sigma = 0.4$).

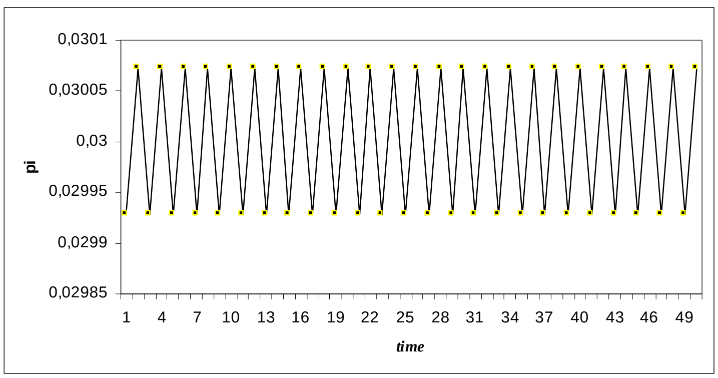

Figure 4. Panel **B**. The effect of a monetary policy change over the inflation rate (long-run impact; $\sigma = 0.4$).

Again, we must stress that the cycles of periodicity 2 are the direct result of assuming a learning process that allows information to spread across the population under such pattern; other gain values can imply fixed-point stability, cycles of periodicity larger than two, or fully irregular cycles. As an illustration recover the panel **B** of section 2. There, chaotic motion is observed for a small range of values of σ. Consider a value of this constant for which chaos effectively holds; e.g., let $\sigma = 0.298$. Figure 5 shows the pattern of evolution for the inflation rate in this specific case - panel **A** characterizes the initial phase of the adjustment process (with $\pi_0 = \Delta m_0 = 0.02$); panel **B** displays the long-run endogenous irregular behavior of the inflation rate.

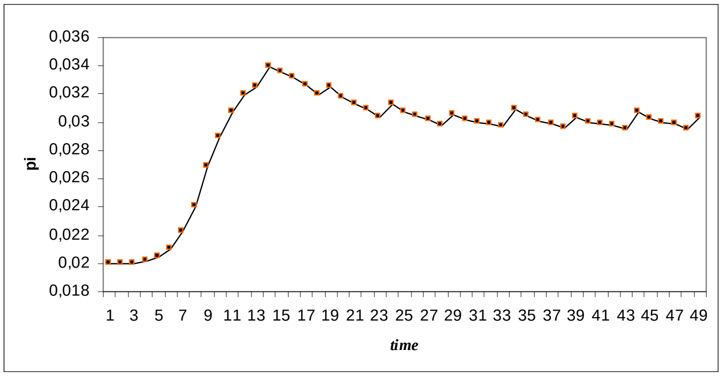

Figure 5. Panel **A**. The effect of a monetary policy change over the inflation rate (first 50 observations; $\sigma = 0.298$).

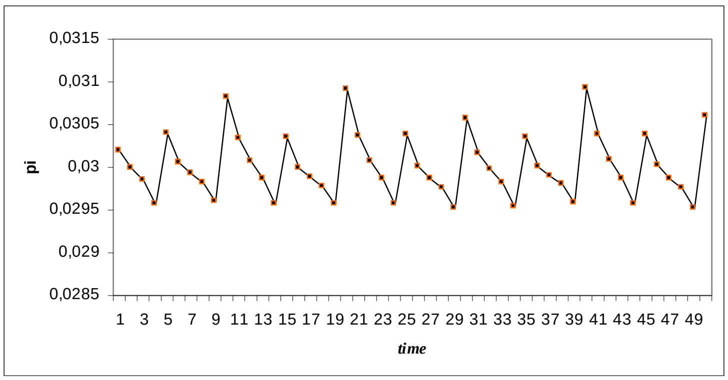

Figure 5. Panel **B**. The effect of a monetary policy change over the inflation rate (long-run impact; $\sigma = 0.298$).

5 Conclusion

Diffusion processes usually explain how some innovation becomes pervasively accepted by the economic system. Social influence may justify an s-shaped adoption curve that culminates in a stable outcome where all the potential adopters end up, sooner or later, gaining access to the new idea, news or technological knowledge. This steady-state fixed-point result can be disturbed when agents fail in perceiving how the potential audience evolves over time; this value can be learned on the aggregate, but misspecification problems may lead to the adoption of an incorrect learning rule. The result is that the steady-state outcome is no longer guaranteed. In the long-run, the share of agents that are potential adopters will systematically change, with some regularity or following completely irregular paths. The systematic change in the share of potential adopters automatically suggests that the long-term share of effective adopters will also suffer the same type of fluctuations. Once we adapt these notions about diffusion to a specific economic setting (e.g., the dissemination of information on a monetary policy change over the universe of price-setting firms), we offer a possible rationale for observed fluctuations in the economic system.

References

[1] Bala, V. and S. Goyal (1998) “Learning from Neighbours.” *Review of Economic Studies* **65**, 595-621.

[2] Barros, P. P. and X. Martinez-Giralt (2009) "Technological Adoption in Health Care." *Universidade Nova de Lisboa and Universitat Autonoma de Barcelona working paper.*

[3] Bass, F. (1969) "A New Product Growth Model for Consumer Durables." *Management Science* **15**, 215-227.

[4] Bass, F. (1980) "The Relationship between Diffusion Rates, Experience Curves and Demand Elasticities for Consumer Durables and Technological Innovations." , *Journal of Business* **53**, 551-567.

[5] Carroll, C. D. (2006) "The Epidemiology of Macroeconomic Expectations." in L. Blaume and S. Durlauf (eds.), *The Economy as an Evolving Complex System III*, Oxford: Oxford University Press.

[6] Chatterjee, K. and S. Hu (2004) "Technology Diffusion by Learning from Neighbors." *Advances in Applied Probability* **36**, 355-376.

[7] Cowan, R. and N. Jonard (2004) “Network Structure and the Diffusion of Knowledge.” *Journal of Economic Dynamics and Control* **28**, 1557-1575.

[8] Geroski, P. A. (2000) "Models of Technology Diffusion." *Research Policy* **29**, 603–625.

[9] Gomes, O. (2009) "Adaptive Learning and Complex Dynamics." *Chaos, Solitons and Fractals* **42**, 1206-1213.

[10] Jackson, M. O. and L. Yariv (2007) “Diffusion of Behavior and Equilibrium Properties in Network Games.” *American Economic Review Papers and Proceedings* **97**, 92-98.

[11] Karshenas, M. and P. Stoneman (1995) "Technological Diffusion." in: Stoneman, P. (Ed.), *Handbook of the Economics of Innovation and Technological Change*. Oxford: Blackwell, 265–297.

[12] Mankiw, N. G. and R. Reis (2002) "Sticky Information versus Sticky Prices: a Proposal to Replace the New Keynesian Phillips Curve." *Quarterly Journal of Economics* **117**, 1295-1328.

[13] Mankiw, N. G. and R. Reis (2006) "Pervasive Stickiness." *American Economic Review* **96**, 164-169.

[14] Mankiw, N. G. and R. Reis (2007) "Sticky Information in General Equilibrium." *Journal of the European Economic Association* **5**, 603-613.

[15] Mukoyama, T. (2006) "Rosenberg’s ’Learning by Using’ and Technology Diffusion." *Journal of Economic Behavior and Organization* **61**, 123-144.

[16] Parente, S. L. (1994) "Technology Adoption, Learning-by-doing, and Economic Growth." *Journal of Economic Theory* **63**, 346-369.

[17] Parente, S. L. (2000) "Learning-by-using and the Switch to Better Machines." *Review of Economic Dynamics* **3**, 675-703.

[18] Gallegati, M.; B. Greenwald; M. G. Richiardi; and J. E. Stiglitz (2008) "The Asymmetric Effect of Diffusion Processes: Risk Sharing and Contagion." *Global Economy Journal* **8**, Issue 3, Article 2.

[19] Young, H. P. (2003) “The Diffusion of Innovations in Social Networks.” in *The Economy as a Complex Evolving System*, vol. III, ed. L. E. Blume and S. N. Durlauf. Oxford: Oxford University Press.

[20] Young, H. P. (2009) "Innovation Diffusion in Heterogeneous Populations: Contagion, Social Influence, and Social Learning." *American Economic Review* **99**, 1899-1924.