Endogenous Growth, Price Stability and Market Disequilibria

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Abstract

Resorting to an endogenous growth framework, the paper studies the implications of taking market clearing as a long term possibility rather than an every period implicit assumption, as in conventional growth analysis. Under the proposed setup, the system may converge to a market equilibrium outcome in the same way it can converge to a state of constant growth; however, local instability may deviate the economy from the long run stable result. The underlying main assumption respects to an adjustment mechanism in which: (i) transitional dynamics are characterized by the persistence of an accumulated market imbalance; (ii) monetary authorities are able to guarantee price stability. The implications of this modelling structure are the following: (a) a market clearing equilibrium may co-exist with other equilibrium points; (b) several types of stability outcomes are obtainable; (c) monetary policy becomes relevant for growth.

Keywords: Endogenous growth, Non-equilibrium models, Keynesian macroeconomics, Local stability analysis, Monetary policy.

JEL classification: O41, C62, E12.

1 Introduction

The modern theory of economic growth studies long term wealth accumulation under the strong assumption that markets are permanently in an equilibrium position. The following sentence, by Cellarier (2006, page 54), clearly states the assumption that is generally implicit in almost all the work involving the analysis of the growth process: 'In equilibrium, total output along with the utilized aggregate level of capital and labor are all equal to their aggregate demand and supply (...). Therefore at any given time period, the market clearing real wage and real interest rate both depend on the current state of the economy'.

To assume that in each time period, from now to an undefined future, the demand level will be persistently equal to the quantity of produced goods can be interpreted as a somehow awkward hypothesis in the context of growth models. After all, noticing that growth setups are essentially intertemporal adjustment frameworks, we understand that the notion of adjustment or convergence sharply contrasts with the conventional assumption that takes markets as automatically and instantaneously in equilibrium.

In this paper, we advocate that rather than being an automatic mechanism, market equilibrium is the result of a lengthy process that initiates in the present moment and that goes on to some point in the future, i.e., we consider the market equilibrium adjustment as an intertemporal phenomenon, just like the accumulation of material wealth. The main goal is to inquire about the consequences of approaching growth in simultaneous with market adjustment and to look at the long term results underlying such an approach. We will understand that the new assumption has a relevant impact over the growth process.

Our focus is on the growth implications of market disequilibria. We can associate this particular point of analysis to the more general debate on the overall role of the absence of instantaneous market clearing on the study of aggregate phenomena. The velocity of market adjustment is far from being a neglected question in macroeconomics; on the contrary, it is in the core of the macroeconomic debate, as Mankiw (2006) points out. The classical tradition understands markets as mechanisms of automatic and instantaneous adjustment; the Keynesian view is one in which disequilibria tends to persist over time, given the inertial nature of markets, often subject to inefficiencies, information problems and coordination failures. It is not surprising, thus, that classical economics has always seemed more prepared to deal with long term growth, while Keynesian economics have focused in the explanation of short run cycles that are triggered by market frictions. The argument is that the analysis of growth is an analysis of long run trends, where one can neglect demand driven features and focus on the structural role of supply side economics. Typically, the classical view is one in which only accumulation of inputs matters for growth. All the discussion about

frictions and the role of demand in shaping the market outcome is left for a short term debate concerning business fluctuations (although even these may be looked at through the lenses of a classical perspective, as the Real Business Cycles theory does).

The 'new Keynesian perspective' [Clarida et. al. (1999)] or 'the new neo-classical synthesis' [Goodfriend and King (1997)], as different authors call the recent attempt to produce a consensus between classics and Keynesians in the macroeconomic analysis, mixes representative agent intertemporal optimization features with inherently Keynesian ideas, associated with nominal sluggishness. It begins to be consensual, in the contemporaneous analysis of macro phenomena, that the best ingredients of both perspectives should be combined, at the same time that the most counterfactual assumptions of the two views are purged, in order to obtain more robust explanations on macro phenomena. The weakest points in the Keynesian analysis have to do with the ad-hoc way in which some aggregate relations are established, while the classical view may be criticized by an excessively optimistic interpretation about market efficiency. From a classical point of view (as we said, the one underlying the main growth paradigms), the synthesis can be interpreted as a requirement to take seriously the factors that impose, in each moment, a lack of coincidence between aggregate demand and aggregate supply (being the main factor involved in this disequilibrium the impossibility of having, in the real world, completely flexible prices).

Besides the overall controversies, the disequilibrium or non-Walrasian approach to macroeconomics has constituted a relevant field of research on its own. Starting with the contributions of Patinkin (1965), Clower (1965), Leijonhufvud (1968), Barro and Grossman (1971), Bénassy (1975) and Malinvaud (1977), demand-supply imbalances and their consequences have been thoroughly debated. The implications of absence of market clearing in the goods market and / or in the labor market have to do essentially with the short run effects of excess demand or excess supply, with the eventual persistence of imbalances in time and with possible solutions to improve market efficiency, for instance, the discussion about the usefulness of a central auctioneer. The type of market structure that arises in face of markets that are not able to adjust automatically is also a matter of concern [e.g., Bénassy (1993, 2002) studies imbalances under a monopolistic competition environment].

Today's macroeconomics continues to address the issue of market disequilibrium in both product and labor markets. Short run economic implications of macro disequilibrium have been addressed, among others, by Flaschel et. al. (1997), Chiarella and Flaschel (2000), Asada et. al. (2003), Chiarella et. al. (2005), Raberto et. al. (2006) and Hallegatte and Ghil (2007). The referred authors use the disequilibrium approach to search for the presence of endogenous business cycles. They introduce market frictions, imperfect rationality in expectations and biases on aggregation as arguments

to produce short run destabilizing effects that analytically translate in bifurcations capable of imposing a transition from a fixed point equilibrium to a region, in the parameters' space, where cycles of increasing periodicity and even completely a-periodic motion is observed.

What the literature tells us is that the study of 'Keynesian equilibria' [a term introduced by Geanakoplos and Polemarchakis (1986) to designate macroeconomic outcomes that deviate from the market clearing result] is, in fact, strongly associated with a short run perspective of the economic system, a perspective in which one may somehow relax the hypotheses of agent optimization, rational expectations and market clearing. Short term economic performance is more likely to be affected by publicly announced government policies and by less than rational behavior and expectations of private agents, in sharp contrast with the understanding one may take of long term trends of growth, which traditionally are viewed as solely determined by the supply side.

There are, nevertheless, good examples of attempts to deal with the long term impact of aggregate demand over growth trends. This is done, for instance, in Palley (1996, 2003), Blackburn (1999) and Dutt (2006). These authors begin precisely by highlighting that conventional macroeconomics only look to the interaction of aggregate demand and aggregate supply in the short run performance of the economy but it neglects the eventual role of demand on the analysis of economic growth: growth is exclusively driven by supply-side factors, like the state of technology. As Dutt (2006, page 319) states: 'for mainstream macroeconomists, aggregate demand is relevant for the short run and in the study of cycles, but irrelevant for the study of growth'; then he asks (page 320): 'is it not more sensible to have a growth theory in which both aggregate supply and aggregate demand considerations have roles to play?'

The main contribution provided by the mentioned authors concerns the eventual impact of aggregate demand over the steady state growth rate of the economy; their theoretical modelling structures point to the possibility of an increased rate of growth as the result of some demand shock, caused for instance by a change on the fiscal policy of the government.

However, the introduction of Keynesian traits into growth setups does not have to change necessarily the long run growth results, typically dominated by supply side determinants. We can continue to accept that aggregate demand has a negligible impact in long run growth trends, but this does not automatically imply that its impact is absent on the transitional dynamic process. Furthermore, we may have long run supply driven results, but aggregate demand considerations may be determinant in shaping stability conditions that are decisive for knowing if the long term steady state is, in fact, accomplished.

This paper departs from the previously mentioned contributions because it relates essentially to the analysis of transitional dynamics. It emphasizes

the idea that an economy starting from a state of non balanced growth and non market equilibrium may converge to (or diverge from) a steady state of balanced growth and market clearing. Thus, the study of stability conditions is central to understand how the economy can attain the desirable long term rate of growth and also how this long term state may be compatible with market efficiency that is absent along the transitional dynamics phase. Therefore, we stress that determinants of long term growth are supply side entities, but the demand side is relevant to understand if such long run equilibrium is attainable and if it reflects an efficient allocation of resources.

In this context, a relevant contribution somehow attached to the problem we propose is furnished by Fanti (2000). This author develops a model of market disequilibrium that generates chaotic price fluctuations. The model considers a market adjustment mechanism in which prices and quantities interact in the generation of the disequilibrium. This is at odds with the setup we develop below because in this last one the disequilibrium is generated solely by mismatches between quantities offered and quantities demanded. Such an assumption is useful because it reduces the dimensionality of the underlying system, allowing for a treatment in the context of conventional classical long term growth models.

It is relevant to stress that in this paper we focus the attention on the goods market. The general equilibrium nature of the presented problem ignores other extremely relevant sources of disequilibria, besides underproduction / overproduction issues. Particularly relevant are the disequilibria in the labor market, that may equally be understood as a source of endogenous cycles in the growth process, as in Fanti and Manfredi (2003).

In the sections that follow, we approach growth by introducing a disequilibrium mechanism in a classical endogenous growth model. We will consider mainly an AK model, as in Rebelo (1991), but the two-sector growth model with human capital, extensively discussed in the literature [e.g., Lucas (1988), Caballé and Santos (1993), Mulligan and Sala-i-Martin (1993), Bond, Wang and Yip (1996), Ladrón-de-Guevara, Ortigueira and Santos (1997), Ramos-Parreno and Sanchez-Losada (2002), Gómez (2003, 2004)], is also developed.

The disequilibrium mechanism is essentially based on a pair of dynamic equations proposed in Hallegatte et. al. (2007) and it is able to eliminate instantaneous market clearing. However, as stated, market clearing is a long term possibility, i.e., in the same way the underlying system may converge to a stable constant growth rate, it can also converge to a market equilibrium result. One of our fundamental underlying hypothesis is the one relating to the efficacy of monetary policy. We treat monetary policy as exogenous to the model, but we adopt the optimistic view that monetary authorities are able to use nominal interest rates in order to maintain price stability: in the long run, the inflation rate will just correspond to the target rate set by the Central Bank. This assumption encounters support in reality, since

it is a clear evidence that developed economies have been able to maintain inflation at relatively stable levels over the last decades; for instance, Woodford (2003, page 2) reminds us that 'since the 1980s the central banks of the major industrial nations have been largely successful at bringing inflation down to low and fairly stable levels'. Because the setup to present is purely deterministic, we will consider a constant inflation rate, neglecting variations around the target rate that even in an economy with stable inflation tend to persist over time.

Three versions of the disequilibrium model will be proposed and their local dynamics will be explored: (i) a one-sector discrete-time setup. In this model, besides the disequilibrium mechanism, another Keynesian feature is present: a constant marginal propensity to consume, which turns consumption into a constant share of output. Afterwards, we extend the model to include a human capital sector; in the two-sector model, we continue to assume absence of any optimization process (there is a constant marginal propensity to consume and the share of human capital in each of the two economic sectors is taken as constant over time and capable of allowing for an eventual market-clearing steady state). A second extension of the model puts it closer to the classical growth analysis by assuming consumption utility intertemporal maximization; the framework continues to assume the transitional dynamics market disequilibrium and the possibility of long run market equilibrium, but the representative agent chooses the level of consumption that maximizes intertemporal utility.

The remainder of the paper is organized as follows. Section 2 presents the basic structure of the model. Section 3 addresses the dynamic behavior of the system. Section 4 introduces the role of human capital by assuming a two-sector endogenous growth model. In section 5 we take an optimal control problem of utility maximization, where the market non equilibrium mechanism is still present. Finally, section 6 is left to conclusions and implications.

2   The Disequilibrium Framework

Consider a closed economy, populated by a large number of agents (households and firms). In this economy, the population level does not grow and it coincides with the available labor; thus, by normalizing the amount of labor to 1, all variables to consider are simultaneously level variables and per capita / per unit of labor variables. Time is defined discretely, $t = 0, 1, \ldots$ and the developed setup will be fully deterministic.

In this economy, growth is endogenous. The adopted definition of endogenous growth is the following:

Definition 1 *All economic aggregates defined in real terms grow, in the steady state, at a constant rate $\gamma \in (-\delta, A - \delta)$. Parameter $\delta \geq 0$ is the*

depreciation rate of capital and $A > 0$ translates the level of technology. The mentioned real aggregates are, for now, the following: $y_t \in \mathbb{R}_+$ (income / output); $k_t \in \mathbb{R}_+$ (physical capital); $d_t \in \mathbb{R}_+$ (demand); $c_t \in \mathbb{R}_+$ (consumption); $i_t \in \mathbb{R}_+$ (irreversible investment); and $x_t \in \mathbb{R}$ [measure of cumulative market imbalances (CMI)].

The last variable in definition 1 requires an explanation. Because in our analysis markets do not clear instantaneously, in each time moment we may have an income level above the level of demand or the opposite. Variable x_t is the sum of such differences from $t = 0$ to the present moment. To weight equally all the output-demand lags, we consider the present value of each one of the lags. Given that economic aggregates (output and demand) grow in time at rate γ, the CMI measure will correspond to

$$x_t = \sum_{i=0}^{t-1} \left[(1+\gamma)^{t-1-i} \cdot (y_i - d_i) \right] + (1+\gamma)^t \cdot x_0 \qquad (1)$$

As defined in expression (1), x_t is the present value of the accumulated market imbalances (given an initial market imbalance x_0). In Hallegate *et. al.* (2007), this variable is called 'goods inventory'. In fact, this is an acceptable term in the sense that it is an entity that is filled with output and emptied with demand. However, the term inventory could be understood as awkward since the assumed accumulated gap may take either positive or negative values. If x_t is positive, it is in fact a physical stock inventories variable; if the variable assumes negative values, it will correspond to the accumulated desire of the demand side in accessing goods that are not being delivered.

The endogenous growth nature of the problem derives from the assumption of an AK constant marginal returns production function, $y_t = Ak_t$. Capital accumulation, in turn, will correspond to the standard difference between investment and capital depreciation, as follows,

$$k_{t+1} - k_t = i_t - \delta k_t, \quad k_0 \text{ given} \qquad (2)$$

The two equations that are essential to characterize the absence of market clearing are withdrawn from the analysis of the goods market in Hallegatte *et. al.* (2007) and they are the following,

$$x_{t+1} - (1+\gamma)x_t = y_t - d_t, \quad x_0 \text{ given} \qquad (3)$$

$$p_{t+1} - (1+\widetilde{\pi})p_t = -\theta p_t \frac{x_t}{d_t}, \quad p_0 \text{ given} \qquad (4)$$

Equation (3) introduces the dynamics of the CMI measure. Note that this equation has, as solution, the one given by expression (1), and thus

equation (3) is just the dynamic translation of the idea that x_t is the present value of the sum of market imbalances. The meaning of the CMI measure in this particular context is understood by interpreting what different signs mean. If $x_t > 0$, a situation of overproduction or a selling lag exists; this positive CMI value has correspondence on the time necessary for firms to sell the produced goods. On the contrary, if $x_t < 0$ then we have a case of underproduction or a delivery lag; the negative CMI corresponds to the time needed for consumers to get the ordered goods. Underproduction is the result of the technical lag associated with the transport and distribution of goods and to the inertia concerning changes in the installed production capacity.

Equation (3) describes the way in which the lack of adjustment between production and demand changes the CMI measure. In the presence of market clearing (i.e., $y_t = d_t$), the CMI will grow at the benchmark growth rate γ (since we are assuming present values); if output exceeds demand the CMI variable grows at a rate above γ; and if the output level is below the demand level the growth rate of the CMI is lower than γ.

Recall that we are working with an endogenous growth setting and we have established that γ will be the long run growth rate of the main economic aggregates, including the CMI measure (if the economy in fact converges to the balanced growth path). In Hallegatte *et. al.* (2007), equation (3) is just $x_{t+1} - x_t = y_t - d_t$, because their model is neoclassical (marginal returns are diminishing) and thus the economy does not grow in the long run. In such case, the CMI grows positively for $y_t > d_t$, negatively for $y_t < d_t$ and it remains unchanged if $y_t = d_t$. In the proposed endogenous growth case, it is fundamental to clarify that market clearing, i.e. $y_t = d_t$, does not imply an unchangeable CMI value; according to (3), market clearing implies that the CMI measure grows at rate γ. Therefore, we establish through the presented equation a relation between the steady state and market clearing: the long term steady state in which variables grow at rate γ is also a state of market clearing.

Equation (4) characterizes price changes as a function of the CMI per unit of demand (variable $p_t \in \mathbb{R}_+$ represents the price level). If the CMI is a positive value, accumulated output exceeds accumulated demand, what makes buyers to concentrate market power and, thus, prices will have a tendency to fall. In the opposite case, the temporary underproduction attributes market power to the supply side, which drives a rise in prices. Parameter $\theta > 0$ measures the degree of sensitivity of prices to changes in the CMI. Parameter $\widetilde{\pi}$ corresponds to the inflation rate that allows for a zero CMI (it can be a positive, zero or negative value).

Defining the change of prices from moment t to moment $t+1$ as the inflation rate in t, π_t, we rewrite equation (4) as the following static relation

$$\pi_t = \widetilde{\pi} - \theta \frac{x_t}{d_t} \qquad (5)$$

In the assumed closed economy with no government intervention at the fiscal level, demand is just given by $d_t = c_t + i_t$. Relatively to consumption, we adopt a simple Keynesian consumption function, $c_t = by_t$, with $b \in (0, 1)$ the marginal propensity to consume. Later, we will model consumption as the optimal result of an intertemporal control problem of utility maximization. We could restrict the analysis to the optimality framework, but as we shall see, it is relevant to observe how the obtained dynamic results change with the assumption of different modelling settings.

Investment is given by the difference between demand and consumption, i.e., $i_t = -\theta \frac{x_t}{\pi_t - \widetilde{\pi}} - by_t$.[1] Because we have assumed that investment is irreversible, the following inequality must hold: $-x_t \geq \frac{b}{\theta}(\pi_t - \widetilde{\pi})y_t$. This inequality gives the relevant information that the CMI variable must have the opposite sign of the term $\pi_t - \widetilde{\pi}$.

The dynamics of the model will be addressed by taking the ratio CMI measure per unit of output, $\varphi_t \equiv x_t/y_t$. According to our definition of steady state, variables x_t and y_t will grow at an identical rate in the steady state and, as a result, the defined ratio will have a constant steady state value. A new equation of motion is presentable by combining equations (2), (3) and (5),

$$\varphi_{t+1} = \frac{1 + (1 + \gamma)\varphi_t + \theta \varphi_t / (\pi_t - \widetilde{\pi})}{1 - \delta - A(b + \theta \varphi_t / (\pi_t - \widetilde{\pi}))} \tag{6}$$

The disequilibrium endogenous growth model can be studied taking into account equation (6) and a rule concerning the motion of the inflation rate. We take monetary policy as exogenous but we assume that the Central Bank undertakes an effective policy aimed at price stability. Through the control of the nominal interest rate, the monetary authority is able to make inflation converge to a previously set inflation rate target, given some initial inflation rate value. Therefore, we take the following equation for the dynamics of inflation,

$$\pi_{t+1} = \varepsilon \pi_t + (1 - \varepsilon)\pi^*, \quad -1 < \varepsilon < 1, \quad \pi_0 \text{ given} \tag{7}$$

Equation (7) indicates that given some parameter ε located inside the unit circle, the inflation will converge from some initial value to its steady state level that is identical to the target set by the monetary authority: $\overline{\pi} = \pi^*$.[2]

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[1] At a first look, one might find odd to take investment as a residual value, i.e., the difference between demand and consumption, with these previously determined by the assumed economic mechanisms. However, this is what one observes as well in conventional growth models, where demand and output are equal by assumption and consumption is derived from some rule or optimality process. Therefore, the hypothesis about the value of investment does not depart from what is typically assumed.

[2] This inflation convergence equation is not an ad-hoc relation. We can obtain it by solving the benchmark New-Keynesian monetary policy model. This problem assumes an

If the target for inflation that is chosen by the Central Bank is the one that allows for a zero CMI value, i.e. $\overline{\pi} = \pi^* = \widetilde{\pi}$, then we are back at the conventional endogenous growth model in which markets clear in every time moment. Interesting dynamics arise if the inflation rate target differs from such inflation value. This can happen if, for instance, the inflation rate allowing for $x_t = 0$ is zero and the Central Bank needs to set a positive inflation target in order to accommodate changes in relative prices, without needing to force negative nominal price changes (e.g., over wages).

Two points about the introduced structure need further clarification. First, we should explain how relations (5) and (7) both apply to price changes. Second, one has to justify exactly how the nominal interest rate and aggregate investment are linked. Equation (5) is an adjustment equation that establishes a relation between the rate of inflation and the measure of CMI. If the Central Bank is able to stabilize inflation, making it converge to a constant value, then equation (5) indicates that market imbalances are influenced by what the monetary authority can achieve (i.e., we identify a real effect of monetary policy). The second point relates to the role of the interest rate. This is a control variable, for the monetary authority, which uses it to primarily attain the goal of price stability. In this context, the interest rate influences investment indirectly through the previously established relation $i_t = -\theta \frac{x_t}{\pi_t - \widetilde{\pi}} - by_t$. A rise in the nominal interest rate allows to lower the rate of inflation and, indirectly, to increase the value of the CMI measure [according to relation (5)]. In the presented investment equation, investment is positively related to π_t and negatively related to x_t; thus, we find the expected relation of opposite sign between the nominal interest rate and investment: when the interest rate rises (falls), the level of investment falls (rises).

Before addressing the stability properties of the model we must reemphasize, based on the previously presented model structure, which is in fact

objective function in which the goal of the Central Bank is to minimize, intertemporally from now to an undefined point in the future, the difference between the observed inflation rate and the chosen target. The monetary policy problem also takes two resource constraints: an IS equation that establishes a relation of opposite sign between the output gap and the real interest rate and a Phillips curve that connects the contemporaneous values of the inflation rate and of the output gap.

By solving this optimal control problem, where the nominal interest rate is the control variable, one finds a two-equation system with the output gap and the inflation rate the endogenous variables. The local analysis of the system implies a saddle-path stability result and (7) will correspond in this case to the saddle-path. Thus, if the Central Bank has the ability to place the system into the stable trajectory, the inflation rate will converge to the target value folowing rule (7) [see Gomes (2007) for a derivation of this relation under the described procedure].

Because inflation dynamics are the result of assuming a New-Keynesian monetary policy optimization problem, we identify two types of Keynesian elements on the model: the ones that allow to obtain a stable inflation from the standard macro optimization framework, and the ones that are present in the persistent market disequilibrium mechanism.

the origin of the considered market disequilibrium. This is the result of monetary policy: the monetary authority sets an inflation target π^* and it is successful in attaining it; if, however, $\pi^* \neq \widetilde{\pi}$, there is a persistence of CMI value at a level different from zero. Therefore, the source of the disequilibrium is the incapacity, the impossibility or the lack of desire (given other policy goals) of choosing an inflation target rate that eliminates market imbalances.

3 Steady State and Stability Analysis

We will be concerned with a specific kind of steady state: the market-clearing steady state. This is defined as follows:

Definition 2 *The market clearing steady state is the long term outcome in which the economy (all the previously defined real variables) grows at the constant rate γ and in which $\overline{y} = \overline{d}$.*

Under the previous definition, the capital constraint (2) can be used to reveal which is in fact the long term growth rate; by imposing $k_{t+1} = k_t$, the result is: $\gamma = (1-b)A - \delta$. The balanced growth path rate of growth is positively dependent on the level of technology and negatively dependent on the marginal propensity to consume and on the depreciation rate.

Hereafter, we assume that equation (6) is compatible with the possibility of long term market clearing, and therefore we rewrite this equation having in consideration the computed long term growth rate,

$$\varphi_{t+1} = \frac{1 + (1 + (1-b)A - \delta)\varphi_t + \theta\varphi_t/(\pi_t - \widetilde{\pi})}{1 - \delta - A(b + \theta\varphi_t/(\pi_t - \widetilde{\pi}))} \tag{8}$$

Consider equations (7) and (8). The first one directly gives the inflation rate target as the inflation rate steady state value. To find the steady state of the CMI - income ratio, we need to apply the following condition to equation (8): $\overline{\varphi} \equiv \varphi_{t+1} = \varphi_t$. Two solutions are encountered, as long as $\pi^* \neq \widetilde{\pi}$: $\overline{\varphi}_1 = -(\pi^* - \widetilde{\pi})/\theta$ and $\overline{\varphi}_2 = -1/A$. The first allows for market clearing; the second implies market clearing in the particular circumstance in which $A = \theta/(\pi^* - \widetilde{\pi})$ (in fact, this case just means that a unique steady state is found; this case is feasible only if the target inflation rate is above the inflation rate allowing for a zero CMI, since the technology level is a positive value). If the second steady state corresponds to the long term state of the economy, the long run scenario will be one of excess demand $(\overline{y} < \overline{d})$ if $A < \theta/(\pi^* - \widetilde{\pi})$, or one of excess supply $(\overline{y} > \overline{d})$ if $A > \theta/(\pi^* - \widetilde{\pi})$. Thus, even though we assume that the representative agent aims at a market clearing equilibrium (that satisfies her needs both as a consumer and as a producer) this may not be attained in the long term if the system converges to the alternative steady state $\overline{\varphi}_2$.

Therefore, it becomes crucial to inquire in which conditions each one of the steady states is stable or unstable. Before proceeding with the stability analysis, however, some remarks about the steady state should be kept in mind:

i) The steady state $\overline{\varphi}_2$ corresponds, always, to a negative CMI; the same is true for $\overline{\varphi}_1$ if the inflation rate target is set above the inflation rate allowing for no accumulated imbalances;

ii) The steady state investment-capital ratio is: $\overline{i}/\overline{k} = (1-b)A$ under $\overline{\varphi}_1$, and $\overline{i}/\overline{k} = \theta/(\pi^* - \widetilde{\pi}) - bA$ under $\overline{\varphi}_2$. Considering the market clearing equilibrium (the first one), $1-b$ becomes the marginal propensity to invest; for the other steady state, the investment-capital ratio is lower than in the market clearing scenario if there is excess supply ($A > \theta/(\pi^* - \widetilde{\pi})$) and the opposite in case of excess demand. Since we have considered irreversible investment, an additional constraint is derived: $A < \theta/[b(\pi^* - \widetilde{\pi})]$;

iii) If there is not market clearing, the economy will grow in the steady state at a rate different from γ. Note that, in the steady state, the growth rate of the CMI measure is: $\frac{\overline{x}_{t+1} - \overline{x}_t}{\overline{x}_t} = \gamma + 1/\overline{\varphi} - \overline{d}/\overline{x}$; in the absence of market clearing, $\overline{\varphi} = \overline{\varphi}_2$ and the demand – CMI ratio may be taken from equation (5); thus, the growth rate of the CMI measure in the steady state is: $\frac{\overline{x}_{t+1} - \overline{x}_t}{\overline{x}_t} = \gamma - A + \theta/(\pi^* - \widetilde{\pi}) = \theta/(\pi^* - \widetilde{\pi}) - bA - \delta$. Since $\overline{\varphi}$ is constant and $\overline{d}/\overline{x}$ is constant, output and demand (as well as consumption and investment) will grow at this rate in the non market clearing equilibrium. This growth rate is above γ if there exists excess demand and it stays below γ under excess supply (this does not mean, however, that a situation of excess demand is preferable to market equilibrium, because although the economy grows more, there is always a part of the household needs that remain unfulfilled given the relative scarcity of supply, i.e., produced goods have to be rationed);

iv) Excess demand and excess supply in the steady state are quantifiable. Note that $\overline{y} = A|\overline{x}|$ and $\overline{d} = (\theta/(\pi^* - \widetilde{\pi}))|\overline{x}|$. In the presence of excess demand ($A < \theta/(\pi^* - \widetilde{\pi})$), the traded quantity is $A|\overline{x}|$, and the excess demand is $(\theta/(\pi^* - \widetilde{\pi}) - A)|\overline{x}| > 0$; in the case of excess supply ($A > \theta/(\pi^* - \widetilde{\pi})$), only the demanded quantity is traded, $(\theta/(\pi^* - \widetilde{\pi}))|\overline{x}|$, and the amount of excess supply is $(A - \theta/(\pi^* - \widetilde{\pi}))|\overline{x}| > 0$.

In what concerns the stability analysis, knowing in anticipation that inflation converges to its target value, independently of real economic conditions, means that we can concentrate on the CMI – output ratio equation. The following derivative is computed:

$$
\begin{aligned}
\left.\frac{\partial \varphi_{t+1}}{\partial \varphi_t}\right|_{(\overline{\varphi},\pi^*)} = & \left\{ \left[1 + (1-b)A - \delta + \frac{\theta}{\pi^* - \widetilde{\pi}} \right] \left[1 - bA - \delta - A\frac{\theta}{\pi^* - \widetilde{\pi}}\overline{\varphi} \right] \right. \\
& + \left. \left[1 + \left(1 + (1-b)A - \delta + \frac{\theta}{\pi^* - \widetilde{\pi}} \right)\overline{\varphi} \right] A\frac{\theta}{\pi^* - \widetilde{\pi}} \right\} \\
& \div \left[1 - bA - \delta - A\frac{\theta}{\pi^* - \widetilde{\pi}}\overline{\varphi} \right]^2
\end{aligned}
$$

For each one of the steady state points:

$$
\left.\frac{\partial \varphi_{t+1}}{\partial \varphi_t}\right|_{(\overline{\varphi}_1,\pi^*)} = \frac{1 - bA - \delta + \frac{\theta}{\pi^* - \widetilde{\pi}}}{1 + (1-b)A - \delta}
$$

$$
\left.\frac{\partial \varphi_{t+1}}{\partial \varphi_t}\right|_{(\overline{\varphi}_2,\pi^*)} = \frac{1 + (1-b)A - \delta}{1 - bA - \delta + \frac{\theta}{\pi^* - \widetilde{\pi}}}
$$

The observed relation $\left.\frac{\partial \varphi_{t+1}}{\partial \varphi_t}\right|_{(\overline{\varphi}_1,\pi^*)} = \left(\left.\frac{\partial \varphi_{t+1}}{\partial \varphi_t}\right|_{(\overline{\varphi}_2,\pi^*)} \right)^{-1}$ implies that if $\overline{\varphi}_1$ is inside the unit circle, $\overline{\varphi}_2$ will be located outside the unit circle and vice-versa. Therefore, the conclusion is that when one of the steady state points is stable, the other one is necessarily unstable (and the opposite). The possible stability cases are synthesized in table 1.

table 1 here

Table 1 indicates the cases in which the market clearing and the non market clearing steady states are attained, given some initial value of the CMI – output ratio. We confirm that despite the desirableness of obtaining a steady state market equilibrium, for some combinations of parameters the economy diverges from such equilibrium and in the direction of a disequilibrium stable situation. The convergence process is determined by only four parameters: the degree of sensitivity of prices to inventory changes, the inflation target, the zero CMI inflation and the technological level.

Figures 1 and 2 depict two alternative cases; the first panel refers to a stable market clearing steady state; the second panel has, as stable steady state, the one that perpetuates the market imbalance. The graphics are drawn for some $\pi_t = \pi^* > \widetilde{\pi}$ (this constraint allows to consider only the two first cases in table 1). In these figures, the bold lines give the position of the function $\varphi_{t+1} = f(\varphi_t)$ in equation (8), for a constant inflation rate. The way in which they intersect the 45° lines determines the type of dynamics; the steady state to the left (the first point of intersection between the two lines) is always the stable one. Note that if $\varphi_0 > -1/A$ in the first figure, or $\varphi_0 > -(\pi^* - \widetilde{\pi})/\theta$ in the second graphic, the system will diverge, being impossible to achieve the stable long term outcome.

figures 1 and 2 here

4   A Two-Sector Growth Model - the Role of Human Capital

In this section, we sophisticate the non equilibrium model by introducing a second productive sector: the education sector.[3] The framework is similar to the one proposed in the literature, but the distribution of human capital across sectors will not be modelled as the result of an optimal choice. This distribution will correspond to the one that allows for a market clearing equilibrium, that is, the share of human capital allocated to the production of goods will now play a similar role to the one that the propensity to consume had in the last section.

Consider the following changes over the non equilibrium model of the previous section. The final goods production function is changed to include a second input, which is human capital, $h_t \in \mathbb{R}_+$. This production function will be of the Cobb-Douglas type, i.e., there are diminishing returns associated to each one of the inputs (physical and human capital) and constant returns to scale (the function is homogeneous of degree one); we define two parameters: $\alpha \in (0,1)$, which corresponds to the output – physical capital elasticity and $u \in (0,1)$, which is the share of human capital used in the production of final goods; the remainder $1-u$, is the share of human capital used as an input in the production of additional human capital. The production function is $y_t = Ak_t^\alpha(uh_t)^{1-\alpha}$. As before, $A > 0$ represents the technology capabilities available at the goods production sector.

The accumulation of human capital obeys the rule

$$h_{t+1} - h_t = g((1-u)h_t) - \delta h_t, \ \ h_0 \text{ given} \tag{9}$$

According to equation (9), the only input of the human capital production function is human capital; the endogenous growth nature of the model is guaranteed by the assumption of a linear production function (i.e., there are constant marginal returns on the accumulation of human capital). Let this function be $g((1-u)h_t) = g_t = B(1-u)h_t$, with $B > 0$ the technology index of the education sector and g_t the output of the production of human capital. To simplify the analysis, it is assumed that human capital depreciates at the same rate δ as physical capital. Besides (9), the dynamic problem in consideration is also composed by equations (2), (3), (5) and (7).

The main difference of the setup in this section relatively to the one-sector case is that although we continue to assume that the disequilibrium (i.e., the lack of instantaneous market clearing) is associated to the final

---

[3]We consider this second sector as an education sector in which human capital is produced. Given the asumption of constant marginal returns in this sector, human capital works as the engine of growth. Besides this, there is no other substantive differences between the two inputs (e.g., any kind of externality deriving from the accumulation of knowledge); thus, we can, alternativelly, interpret the production factors simply as capital input 1 and capital input 2.

goods sector, now we transfer the source of endogenous growth to a second sector – the education sector. Observe that it is straightforward to realize that human capital grows at a constant rate, independently of the time moment, $\frac{h_{t+1}-h_t}{h_t} = B(1-u) - \delta$.

If one defines the steady state as in the last section (all real variables grow at a same constant rate in the steady state), the presented growth rate is also the growth rate of physical capital, final goods output, demand, consumption and investment. Furthermore, if one continues focused on the idea of a market clearing steady state, this implies that the referred growth rate is equal to the parameter in the CMI dynamic equation, i.e., $\gamma = B(1-u) - \delta$. The steady state growth rate of the economy rises with a better human capital technology level, with additional human capital allocated to the education sector and with a fall in the rate of depreciation.

Consider the following ratios: $\phi_t \equiv x_t/g_t$ and $\omega_t \equiv k_t/h_t$. These allow to obtain the system of equations that follows,

$$\phi_{t+1} = \frac{\frac{Au^{1-\alpha}}{B(1-u)}\omega_t^\alpha + \left[1 + B(1-u) - \delta + \frac{\theta}{\pi^* - \widetilde{\pi}}\right]\phi_t}{1 + B(1-u) - \delta} \qquad (10)$$

$$\omega_{t+1} = \frac{(1-\delta)\omega_t - \frac{\theta}{\pi^* - \widetilde{\pi}}B(1-u)\phi_t - bAu^{1-\alpha}\omega_t^\alpha}{1 + B(1-u) - \delta} \qquad (11)$$

Note that we have already replaced, in (10) and (11), the growth rate parameter by the corresponding expression. From the previous two equations, one obtains a unique steady state value for each one of the endogenous quotients: $\overline{\omega} = \left[\frac{(1-b)A}{B(1-u)}\right]^{1/(1-\alpha)} u$ and $\overline{\phi} = -\left[\frac{A}{B(1-u)}\right]^{1/(1-\alpha)} (1-b)^{\alpha/(1-\alpha)} u \frac{\pi^* - \widetilde{\pi}}{\theta}$.[4] Thus, a unique steady state exists. Observe that $\overline{\varphi} = \overline{\phi}\frac{B(1-u)}{Au^{1-\alpha}\overline{\omega}^\alpha} = -\frac{\pi^* - \widetilde{\pi}}{\theta}$, that is, the obtained steady state is the one allowing for market clearing, with market clearing implying the same long term CMI – output ratio as in the one sector model.

The main difference relatively to the one sector model is that because the growth rate is not determined in the disequilibrium sector, the second steady state (which did not allow for market clearing) no longer exists and, thus, our unique concern becomes to inquire about the stability of the obtained market clearing steady state.

The linearization of the system in the steady state vicinity yields:[5]

_____

[4]Note that the steady state result for ϕ_t is in accordance with the constraint one has previously imposed on the sign of the CMI measure in the steady state, i.e., the CMI (and, thus, variable ϕ_t) is a negative value if $\pi^* > \widetilde{\pi}$ (it will be a positive value if the symmetric condition holds).

[5]Being the dynamics of the inflation rate exogenous to our model, we can analyze local dynamics taking as endogenous variables solely the two assumed quotients (the CMI - human capital output ratio and the physical capital - human capital ratio).

$$\left[\begin{array}{c} \phi_{t+1} - \overline{\phi} \\ \omega_{t+1} - \overline{\omega} \end{array} \right] =$$

$$\left[\begin{array}{cc} 1 + \dfrac{\theta/(\pi^* - \widetilde{\pi})}{1 + B(1-u) - \delta} & \dfrac{\alpha}{(1-b)[1 + B(1-u) - \delta]} \\ -\dfrac{\theta}{\pi^* - \widetilde{\pi}} \dfrac{B(1-u)}{1 + B(1-u) - \delta} & \dfrac{(1-b)(1-\delta) - \alpha b B(1-u)}{(1-b)[1 + B(1-u) - \delta]} \end{array} \right] \cdot \left[\begin{array}{c} \phi_t - \overline{\phi} \\ \omega_t - \overline{\omega} \end{array} \right] \qquad (12)$$

The trace and determinant of the Jacobian matrix in (12) are, respectively,

$$Tr(J) = 1 + \frac{(1-b)(1 - \delta + \theta/(\pi^* - \widetilde{\pi})) - \alpha b B(1-u)}{(1-b)[1 + B(1-u) - \delta]}$$

$$Det(J) = \frac{(1-b)(1-\delta) - \alpha b B(1-u)}{(1-b)[1 + B(1-u) - \delta]} + \frac{\theta}{\pi^* - \widetilde{\pi}} \frac{1 + \alpha B(1-u) - \delta}{[1 + B(1-u) - \delta]^2}$$

Trace and determinant expressions tell us that a positive ratio $\frac{\theta}{\pi^* - \widetilde{\pi}}$ requires $Tr(J) - 1 > Det(J)$, i.e., in the circumstance in which $\pi^* > \widetilde{\pi}$, stability (two eigenvalues inside the unit circle) is not a feasible result [stability condition $1 - Tr(J) + Det(J) > 0$ is not verified]. Stability may arise only in cases in which the Central Bank sets the inflation rate target below the inflation rate that allows for a zero CMI value.

Replacing the ratio $\theta/(\pi^* - \widetilde{\pi})$ in the determinant expression by the corresponding value in terms of the trace, one obtains $Det(J) = v + \frac{1 + \alpha B(1-u) - \delta}{1 + B(1-u) - \delta} Tr(J)$, with v a combination of parameters b, B, u, α and δ. The previous expression indicates the presence of a determinant – trace relation with a positive but lower than one slope. Such expression may be represented graphically, in a trace-determinant diagram, in order to assess the stability properties of the system (figure 3).

figure 3 here

In figure 3, the several lines, $1 - Tr(J) + Det(J) = 0$, $1 + Tr(J) + Det(J) = 0$ and $1 - Det(J) = 0$, represent lines of bifurcation. The inverted triangle formed by these lines corresponds to the area of stability. To the right and the left of the triangle, the system is characterized by saddle-path stability, while instability prevails if $1 - Det(J) < 0$ or, in simultaneous, $1 - Tr(J) + Det(J) < 0$ and $1 + Tr(J) + Det(J) < 0$. The bold line respects to the eventual location of the two-sector model dynamics. All possible local dynamic results are attainable (stability, saddle-path stability and instability), depending on parameter values. The points of bifurcation are quantifiable, and their computation allows us to present analytically the stability result:

1. for $\pi^* > \widetilde{\pi}$:

(a) Instability: $\pi^* - \widetilde{\pi} < \frac{\theta(1-b)[1+\alpha B(1-u)-\delta]}{[1-b(1-\alpha)]B(1-u)[1+B(1-u)-\delta]}$;

(b) Saddle-path stability: $\pi^* - \widetilde{\pi} > \frac{\theta(1-b)[1+\alpha B(1-u)-\delta]}{[1-b(1-\alpha)]B(1-u)[1+B(1-u)-\delta]}$.

2. for $\pi^* < \widetilde{\pi}$:

(a) Stability: $\widetilde{\pi} - \pi^* > \frac{\theta(1-b)[2(1-\delta)+(1+\alpha)B(1-u)]}{2[2(1-\delta)+(1-b(1-\alpha))B(1-u)][1+B(1-u)-\delta]}$;

(b) Saddle-path stability: $\widetilde{\pi} - \pi^* < \frac{\theta(1-b)[2(1-\delta)+(1+\alpha)B(1-u)]}{2[2(1-\delta)+(1-b(1-\alpha))B(1-u)][1+B(1-u)-\delta]}$.

The presented conditions indicate that stability requires setting the inflation rate target below the one that allows for a zero steady state CMI level. Therefore, in the present model, stability implies a positive long term CMI value (recall that an inflation target below $\widetilde{\pi}$ implies $\overline{x} > 0$).

In synthesis, the developed two-sector model that involves a disequilibrium mechanism, in which there is two forms of capital and where no control variable is considered, has a unique market clearing steady state. This steady state is a stable point if $\pi^* < \overline{\pi}$ and if the difference between the benchmark inflation rate and the corresponding target is a value superior to some combination of parameters (the one given in the above inequality regarding the stable case). Thus, in what concerns monetary policy, in the present case the Central Bank guarantees convergence to a balanced growth path in which market clearing also holds if the inflation target rate set by the Central Bank is a value significantly below $\widetilde{\pi}$.

Saddle-path stability may also mean convergence to the steady state, however this would require the initial quantities of physical and human capital to be such that the one-dimensional stable trajectory had to be followed. Thus, to guarantee stability, public authorities may replace a monetary policy aimed at a possibly exaggerate low inflation rate by a policy that encounters the perfect equilibrium between the quantities of physical and human capital. This balance is not easy to accomplish because physical capital and human capital are both state variables. Only indirectly may the government succeed in generating the physical-human capital ratio that locates over the stable arm; this can be done by changing incentives to the use of households savings from physical investment towards education or the other way around.[6]

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[6]By varying parameters' values, namely the difference $\widetilde{\pi} - \pi^*$, we have observed, through figure 3, that two bifurcation points are crossed: a flip bifurcation occurs in the transition from the region of saddle-path stability to the region of stability, and a Neimark-Sacker bifurcation characterizes the passage from saddle-path stability to instability. By using various numerical examples, we have searched for nonlinear dynamics close to these bifurcation points. No periodic or a-periodic cycles were found. Only a perfect coincidence between the local dynamics we have just characterized and the effectively observed global dynamics of the system.

5 Getting Closer to the Classics: Non-Equilibrium Intertemporal Optimization

In the previous sections, one has assumed that the representative agent does not optimize consumption or the distribution of inputs across sectors in order to obtain the best feasible intertemporal utility of consumption. However, non optimization is not a pre-requisite of the non market clearing analysis. In this section, we develop a one sector model similar to the ones in sections 2 and 3 where, nevertheless, the constant marginal propensity to consume is replaced by a process of intertemporal utility maximization.

Let the representative consumer maximize

$$U_t = \sum_{t=0}^{+\infty} [\beta^t u(c_t)] \tag{13}$$

Parameter $\beta \in (0,1)$ is the discount factor and $u(c_t)$ is the instantaneous utility function. We consider a simple continuous and differentiable utility function with diminishing marginal consumption utility of the logarithmic form, $u(c_t) = ln(c_t)$. The maximization of U_0 is subject to resource constraints (2) and (3) and to the inflation equations (5) and (7). The main distinctive feature relatively to the problem of section 2, is that now consumption is not a fixed proportion of income, but the result of an optimization process.

Besides consumption, it is reasonable to assume that inflation is also a control variable of the problem. The private economy is, after all, the first responsible by determining price changes. However, recall that the proposed framework gives a determinant role to the monetary authority in what concerns price setting. These two assumptions are not incompatible under the proposed setup. From the point of view of the maximization problem, considering inflation as a control variable will not impose any rule for the evolution of this variable in time; it will just simplify the expression relating to the motion of the consumption aggregate.

The Hamiltonian function of the problem is:

$$
\begin{aligned}
H(y_t, x_t, c_t, \pi_t, p_t^y, p_t^x) &= u(c_t) - \beta p_{t+1}^y \left[A \left(\frac{\theta}{\pi_t - \widetilde{\pi}} x_t + c_t \right) + \delta y_t \right] \\
&\quad + \beta p_{t+1}^x \left[y_t + \left(\gamma + \frac{\theta}{\pi_t - \widetilde{\pi}} \right) x_t \right]
\end{aligned}
$$

$p_t^y, p_t^x \in \mathbb{R}$ are the co-state variables of y_t and x_t, respectively.
First-order optimality conditions are:

$$H_c = 0 \Rightarrow 1/c_t = A\beta p_{t+1}^y$$

$$H_\pi = 0 \Rightarrow A p_{t+1}^y = p_{t+1}^x$$

$$\beta p_{t+1}^y - p_t^y = -H_y \Rightarrow (1-\delta)\beta p_{t+1}^y - p_t^y = -\beta p_{t+1}^x$$

$$\beta p_{t+1}^x - p_t^x = -H_x \Rightarrow \left(1 + \gamma + \frac{\theta}{\pi_t - \widetilde{\pi}}\right)\beta p_{t+1}^x - p_t^x = A\frac{\theta}{\pi_t - \widetilde{\pi}}\beta p_{t+1}^y$$

$$\lim_{t \to +\infty} y_t \beta^t p_t^y = 0 \quad \text{(transversality condition)}$$

$$\lim_{t \to +\infty} x_t \beta^t p_t^x = 0 \quad \text{(transversality condition)}$$

From the first three optimality conditions, it is obtained a constant growth rate for consumption:

$$\frac{c_{t+1} - c_t}{c_t} = \beta(1 + A - \delta) - 1 \tag{14}$$

The dynamic analysis will proceed by recovering the CMI – output ratio of section 2, φ_t, and by defining the consumption – output ratio $\psi_t \equiv c_t/y_t$. Once more, the steady state is defined as the state in which all real variables grow at a same constant rate and, thus, $\overline{\varphi}$ and $\overline{\psi}$ are constant values.

A pair of difference equations describes the dynamics of the problem under study,

$$\varphi_{t+1} = \frac{1 + (1+\gamma)\varphi_t + \theta\varphi_t/(\pi_t - \widetilde{\pi})}{1 - \delta - A[\theta\varphi_t/(\pi_t - \widetilde{\pi}) + \psi_t]} \tag{15}$$

$$\psi_{t+1} = \frac{\beta(1 + A - \delta)}{1 - \delta - A[\theta\varphi_t/(\pi_t - \widetilde{\pi}) + \psi_t]}\psi_t \tag{16}$$

The definition of steady state implies that the steady state growth rate must be the one presented in equation (14). The definition of market clearing steady state requires, as seen in section 3, that the long run growth rate is γ. Hence, for the case in appreciation, since one wants to discuss the possibility of convergence to market clearing, it is true that $\gamma = \beta(1 + A - \delta) - 1$. The steady state growth rate of the economy under long term market clearing is as much higher as the higher is the value of the discount factor (the lower is the intertemporal discount rate), the better are the technological capabilities and the lower is the capital depreciation rate.

Replacing γ in system (15)-(16), we get a unique solution $\overline{\varphi} = -(\pi^* - \widetilde{\pi})/\theta$, $\overline{\psi} = \frac{1-\beta}{A}(1 + A - \delta)$. This steady state point allows for market clearing. The steady state CMI - income ratio is the same obtained in previous occasions: the opposite sign relation between the CMI measure and $\pi^* - \widetilde{\pi}$ continues to be a central feature of the proposed theoretical structure.

The stability of the equilibrium point is now addressed. In the vicinity of $(\overline{\varphi}, \overline{\psi})$, the system takes the linearized form:

$$
\begin{bmatrix} \varphi_{t+1} - \overline{\varphi} \\ \psi_{t+1} - \overline{\psi} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\theta/(\pi^*-\widetilde{\pi})-A}{\beta(1+A-\delta)} & -\frac{A(\pi^*-\widetilde{\pi})/\theta}{\beta(1+A-\delta)} \\ \frac{1-\beta}{\beta}\frac{\theta}{\pi^*-\widetilde{\pi}} & \frac{1}{\beta} \end{bmatrix} \cdot \begin{bmatrix} \varphi_t - \overline{\varphi} \\ \psi_t - \overline{\psi} \end{bmatrix} \qquad (17)
$$

The trace and the determinant of the Jacobian matrix in (17) are, respectively, $Tr(J) = \frac{1+\beta}{\beta} + \frac{\theta/(\pi^*-\widetilde{\pi})-A}{\beta(1+A-\delta)}$; $Det(J) = \frac{1}{\beta} + \frac{\theta/(\pi^*-\widetilde{\pi})-\beta A}{\beta^2(1+A-\delta)}$.

Stability conditions come,

$$
1 + Tr(J) + Det(J) = 2\frac{1+\beta}{\beta} + \frac{1}{\beta(1+A-\delta)}\left(\frac{1+\beta}{\beta}\frac{\theta}{\pi^*-\widetilde{\pi}} - 2A\right) > 0
$$

$$
1 - Tr(J) + Det(J) = \frac{1-\beta}{\beta}\frac{1}{\beta(1+A-\delta)}\frac{\theta}{\pi^*-\widetilde{\pi}} > 0
$$

$$
1 - Det(J) = -\frac{1-\beta}{\beta} - \frac{\theta/(\pi^*-\widetilde{\pi})-\beta A}{\beta^2(1+A-\delta)} > 0
$$

The second condition is satisfied for any positive $\pi^* - \widetilde{\pi}$. This gives us a first clue on stability results: stability, once again defined as the case in which both eigenvalues of the Jacobian matrix of the system lie inside the unit circle, requires a negative value of the CMI measure, a result that is precisely the opposite relatively to the one we have encountered in the two-sector non-optimization problem.

A diagram drawn in the trace-determinant referential will allow to understand under which conditions different types of stability prevail. For different values of $\frac{\theta}{\pi^*-\widetilde{\pi}}$ the relation between trace and determinant is $Det(J) = -\frac{1+\beta A-\delta}{\beta^2(1+A-\delta)} + \frac{1}{\beta}Tr(J)$. When the constraint $\frac{\theta}{\pi^*-\widetilde{\pi}} > 0$ holds, then we must have: $Tr(J) > 1 + \frac{1-\delta}{\beta(1+A-\delta)}$ and $Det(J) > \frac{1-\delta}{\beta(1+A-\delta)}$. Figure 4 presents a line that characterizes the possible stability outcomes.

figure 4 here

Figure 4 reveals the presence of stability, saddle-path stability and instability for different combinations of parameter values. To present in a systematic way these results, we proceed as in the previous section and separate the cases in which $\pi^* > \widetilde{\pi}$ and the ones in which $\pi^* < \widetilde{\pi}$. As the figure shows, the first ones may correspond to stability or instability, while the seconds will correspond to saddle-path stability or instability. Stability results are:

1. for $\pi^* > \widetilde{\pi}$:

    (a) Instability: $\pi^* - \widetilde{\pi} < \frac{\theta}{\beta[\beta A - (1-\beta)(1-\delta)]}$;

    (b) Stability: $\pi^* - \widetilde{\pi} > \frac{\theta}{\beta[\beta A - (1-\beta)(1-\delta)]}$.

2. for $\pi^* < \widetilde{\pi}$:

    (a) Instability: $\widetilde{\pi} - \pi^* < \frac{\theta}{2\left[\frac{\beta 2}{1+\beta}A + \beta(1-\delta)\right]}$;

    (b) Saddle-path stability: $\widetilde{\pi} - \pi^* > \frac{\theta}{2\left[\frac{\beta 2}{1+\beta}A + \beta(1-\delta)\right]}$.

The previous results confirm that full stability requires setting the inflation rate target above $\widetilde{\pi}$ (and the difference between these two rates above a given threshold value). Saddle-path stability is present for $\widetilde{\pi} - \pi^* > 0$ and as long as the difference between the two inflation rates is kept above the value of the presented combination of parameters. In this framework, in which the representative agent controls the level of consumption, in order to maximize utility, saddle-path stability is a relevant stability result because the agent may choose a consumption level that puts the system over the one-dimensional stable trajectory. Recall that stability will imply two important long term achievements: utility maximization and market clearing.[7]

5.1   A Numerical Example

To illustrate the obtained results, we consider a numerical example. Given the similarities in the dynamic process among the three presented growth problems (in the sense that they all allow for some kind of stability under specific inflation targets), we restrict the application of the example to the optimization problem of this section. The values of the discount factor and of the depreciation rate are withdrawn from the calibration of a macro-model in Guo and Lansing (2002): $\beta = 0.962$ and $\delta = 0.067$. The value of the parameter in equation (3) is the one in Hallegatte *et. al.* (2007), i.e., $\theta = 0.0036$. The technology index is set in order to have a value that allows for a reasonable steady state growth rate; letting $\gamma = 0.05$, then, $A = (1+\gamma)/\beta - (1-\delta) = 0.1585$. Relatively to the parameter in the inflation difference equation (7), we take $\varepsilon = 0.9$ (the value of this parameter is not specially relevant, as long as it stays inside the unit circle, as assumed). We consider as well that the inflation rate needed to attain the absence of market imbalances is zero: $\widetilde{\pi} = 0$.

---

[7]As in the two-sector framework developed earlier, the bifurcations identified in the present situation do not produce any cyclical perpetual motion (however, the Neimark-Sacker bifurcation that separates the regions of stability and instability for $\pi^* > \widetilde{\pi}$, gives place to extremely long transient phases where cyclical motion is observed, once we enter the instability area).

The numerical example serves to show that, as long as stability holds, the transitional dynamics are characterized by a process of simultaneous convergence to a long term growth rate and to a long term market clearing result. The analytical treatment of the model implied a constraint on the value of the inflation rate target in order to guarantee stability. For the chosen parameter values, this constraint is $\pi^* > 0.032$, that is, the monetary authority has to impose an inflation target above 3.2% if it wants the constant growth steady state to be accomplished. Otherwise, the model just diverges from equilibrium (in particular, the consumption – capital ratio falls to zero); if the system rests over the bifurcation line, i.e., if the inflation rate target is 3.2%, then perpetual cycles around the steady state will be evidenced.

The numerical example is graphically illustrated with figures 5 and 6, for an inflation target that guarantees stability ($\pi^* = 0.034$). The figures correspond to the representation of the growth rate of output and of the demand-output ratio, respectively, from an initial point $(\varphi_0, \psi_0, \pi_0)=(-9, 0.5, 0.05)$ till the observation $1,000$.

figures 5 and 6 here

Figure 5 shows that the economy oscillates around a constant steady state growth rate of 5% and that it tends to it in the long run. Along with the growth process, there is a process of market convergence that leads the system in the direction of a long term market clearing state, where $\overline{y} = \overline{d}$, as revealed in figure 6; the system oscillates around the market clearing result but it will rest over that state only in the long term. The process of convergence will be as faster as the larger is the value of the inflation rate target (relatively to $\widetilde{\pi}$).

Finally, we present figure 7, that respects to the behavior of ratio φ_t. In the present example, the CMI measure is systematically negative and it oscillates, with cycles of decreasing periodicity, towards the long run steady state $\overline{\varphi} = -(\pi^* - \widetilde{\pi})/\theta = -9.444$.

figure 7 here

6   Policy Implications and Discussion

Mainstream growth theory, including endogenous growth models, has always involved a time paradox; while the process of resource accumulation and generation of wealth is subject to an evolution from an initial state to a steady state, no similar process is found in what concerns market adjustment. Market equilibrium is instantaneous and no place is left for a dynamic transition from an initial state of excess demand or excess supply to a market clearing outcome.

The possibility of market adjustment is easily introduced by considering a measure of disequilibrium that is filled with increased output and emptied with increased demand. Establishing, then, a relation between the disequilibrium measure and the growth of the price level, the framework of simultaneous growth and market disequilibrium becomes ready to the dynamic analysis. The behavior of the inflation rate can be understood, following the observed reality in modern economies, as the strict result of a monetary policy that uses the nominal interest rate as an instrument to directly determine the evolution of the price level.

The non equilibrium endogenous growth model was analyzed under three different settings. First, a simple consumption function, in which consumption is a constant share of the income level, was considered. This allowed addressing dynamics under a one-dimensional difference equation. Assuming that the representative agent aims at a market clearing steady state, one observes that there are two steady state points; one of them guarantees long term market clearing alongside with endogenous growth at a given rate, while the other leads to situations of permanent excess demand or excess supply. The study of stability indicates that the two points represent different stability outcomes (when one is stable, the other is unstable); in particular, the market clearing steady state prevails when the inflation rate target set by the Central Bank is bounded in a given interval (the boundaries of this interval are dependent on the technology index of the production function, on the depreciation rate of capital, on the marginal propensity to consume and on the elasticity between inflation and the measure of disequilibrium per unit of demand). In this way, we understand the relevant role of monetary policy over growth, which is absent in conventional growth models; the Central Bank may change the inflation target in order to guarantee a situation of long term market equilibrium (a situation where long term growth is compatible with a coincidence of interests between demand and supply agents).

On a second stage, an education sector was introduced. This has changed significantly the dynamics of the system because while the disequilibrium continues associated with the final goods sector, the source of endogenous growth is now linked to the linear shape of the human capital production function. A unique steady state exists when the representative agent searches for a market clearing equilibrium. The stability of this steady state point requires an important monetary policy measure: to set the inflation rate target at a lower value than the inflation rate that allows for a zero accumulated disequilibrium value. The difference between these two rates should be set at a value above a given combination of parameters that involves the inflation – market disequilibrium sensitivity parameter, the capital depreciation rate, the marginal propensity to consume, the output – physical capital elasticity, the share of human capital allocated to the production of physical goods and the education sector technology (but not the goods sector

technology). This relatively low value of the inflation target guarantees the existence of stable market clearing steady state.

Finally, a third model has abandoned the simple linear consumption function, replacing it by an optimization behavior of the private economy representative agent. This controls, as a household, the level of consumption and, as a firm, the way prices evolve over time. With these two control variables, the agent maximizes consumption utility. Once again, we have looked at the eventual existence of a market clearing steady state. It exists and it is unique. The analysis of local dynamics allows to find stability for relatively high values of the inflation rate target. Here, a low inflation rate target is advantageous because it allows for a not too negative steady state level of the disequilibrium measure (i.e., the delivery lag is straightened if price increases are kept low); nevertheless, it can lead to an unstable outcome and therefore a divergence from the market clearing endogenous growth steady state.

The conclusion is that if one departs from the strong assumption of instantaneous market clearing when analyzing growth processes, monetary policy becomes relevant since price changes are an important influence over the real economy. Monetary policy should be such that the selected interest rate rule allows for a convergence to an inflation target. In turn, the inflation target must be low (to avoid cases of too high instantaneous underproduction) but not so low that it prevents convergence to the desired long run state.

A final point of interest concerns the comparison of the proposed setup with the basic AK growth model with instantaneous market clearing. This has plain and straightforward transitional dynamics (in fact, in the simple non optimization version with an exogenous saving rate the AK model possesses no transitional dynamics); typically, when we take an utility intertemporal maximization setup, the endogenous growth model generates a saddle-path stability outcome, independently of parameter values. Thus, the assumption of a market adjustment process alongside with the growth process introduces the possibility of a wide and rich set of dynamic possibilities otherwise absent and in which, as seen in the figures of the numerical example of section 5, periods of excess demand alternate with periods of excess supply, producing sequential phases of low growth and high growth (relatively to the long run rate).

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8   Tables and figures

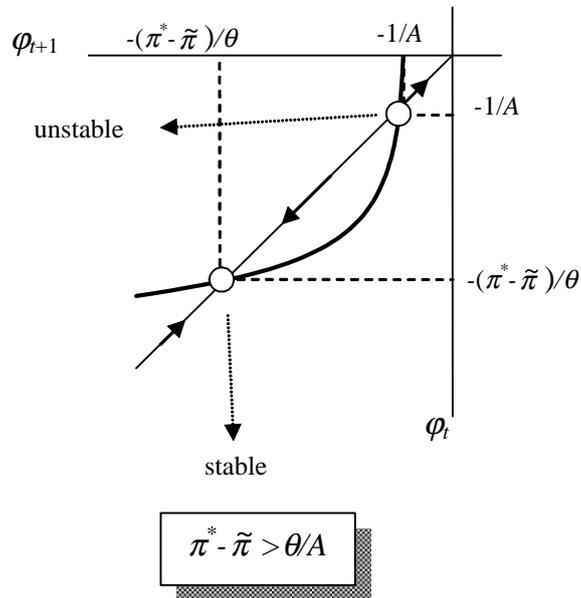| Cases	Stable steady state	Market clearing
$\begin{cases} A > \frac{2(1-\delta)}{2b-1} \\ \pi^* - \widetilde{\pi} > \frac{\theta}{A} \\ \wedge \\ \pi^* - \widetilde{\pi} < \frac{\theta}{(2b-1)A-2(1-\delta)} \end{cases}$	$\overline{\varphi}_1$	Yes
$\begin{cases} A > \frac{2(1-\delta)}{2b-1} \\ \pi^* - \widetilde{\pi} < \frac{\theta}{A} \\ \vee \\ \pi^* - \widetilde{\pi} > \frac{\theta}{(2b-1)A-2(1-\delta)} \end{cases}$	$\overline{\varphi}_2$	No $\begin{cases} \cdot \overline{y} < \overline{d} \text{ if} \\ \pi^* - \widetilde{\pi} < \frac{\theta}{A}; \\ \cdot \overline{y} > \overline{d} \text{ if} \\ \pi^* - \widetilde{\pi} > \frac{\theta}{(2b-1)A-2(1-\delta)} \end{cases}$
$\begin{cases} A < \frac{2(1-\delta)}{2b-1} \\ \pi^* - \widetilde{\pi} < \frac{\theta}{(2b-1)A-2(1-\delta)} \\ \vee \\ \pi^* - \widetilde{\pi} > \frac{\theta}{A} \end{cases}$	$\overline{\varphi}_1$	Yes
$\begin{cases} A < \frac{2(1-\delta)}{2b-1} \\ \pi^* - \widetilde{\pi} > \frac{\theta}{(2b-1)A-2(1-\delta)} \\ \wedge \\ \pi^* - \widetilde{\pi} < \frac{\theta}{A} \end{cases}$	$\overline{\varphi}_2$	No $\overline{y} < \overline{d}$

Table 1 - Steady state stability

Figure 1: Stability of the market clearing steady state in the constant propensity to consume one-sector model.
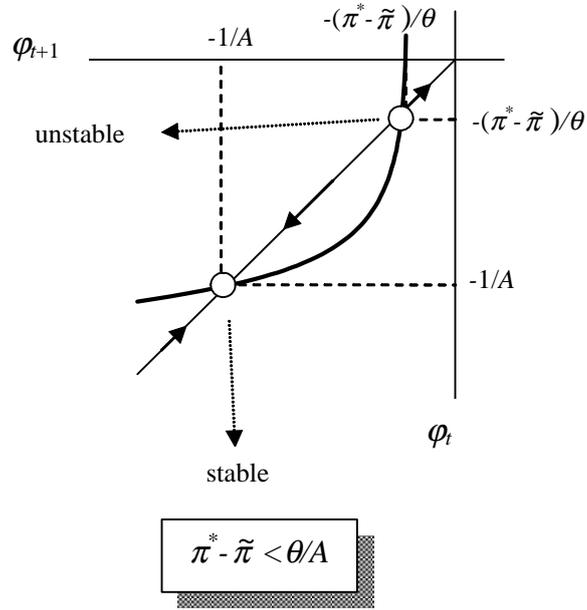
Figure 2: Stability of the non market clearing steady state in the constant propensity to consume one-sector model.
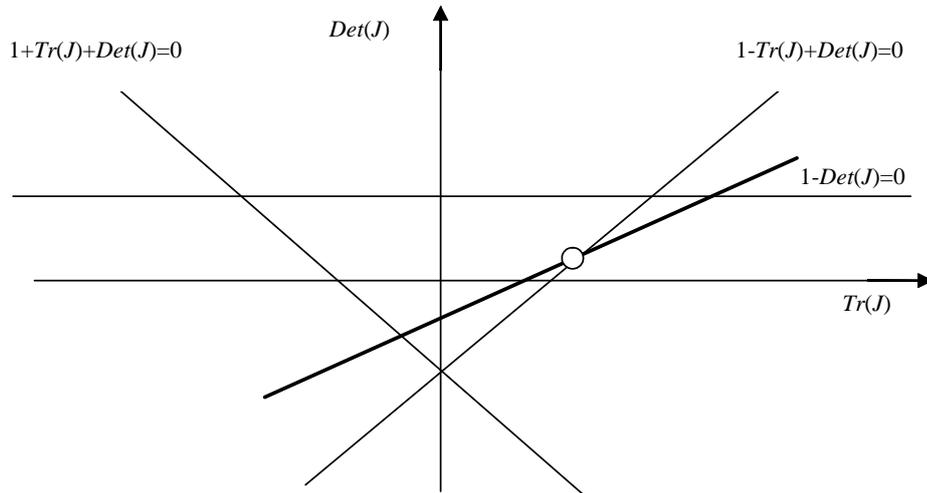
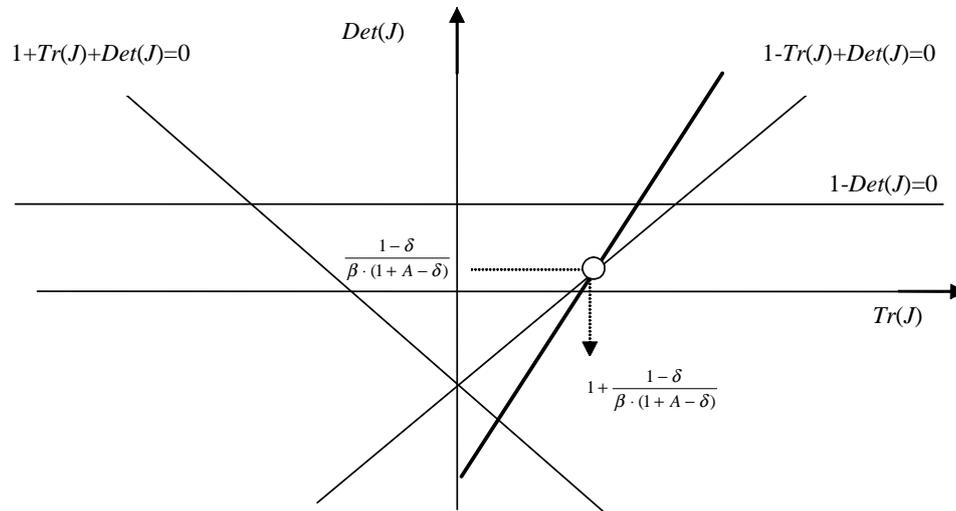Figure 3: Trace-determinant diagram in the two-sector model.

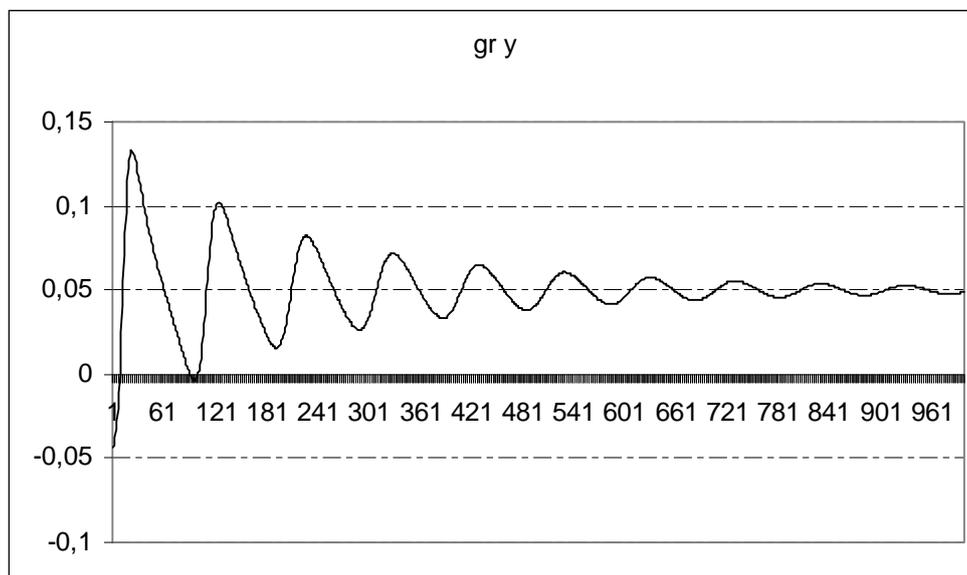Figure 4: Trace-determinant diagram in the optimization problem.

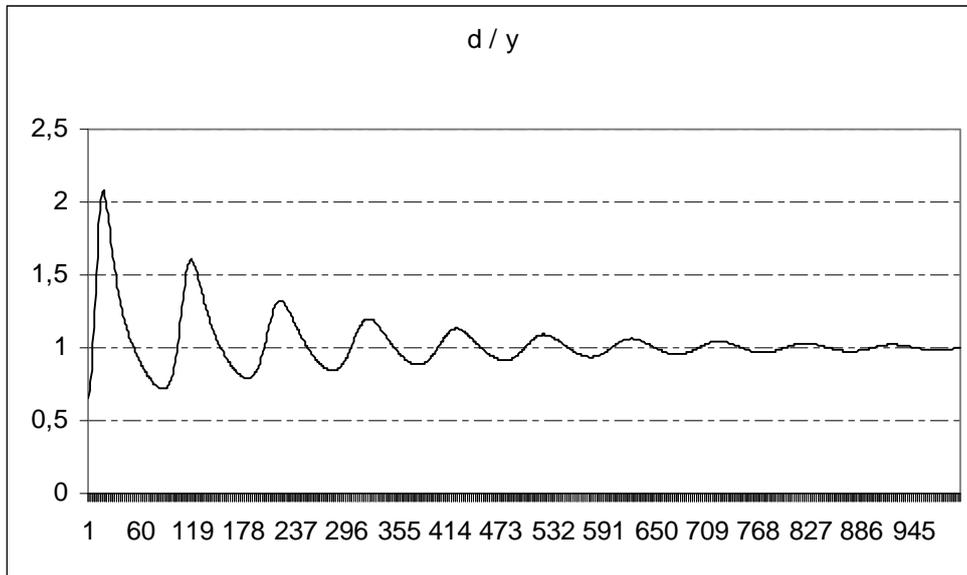Figure 5: Numerical example: output growth rate dynamics.
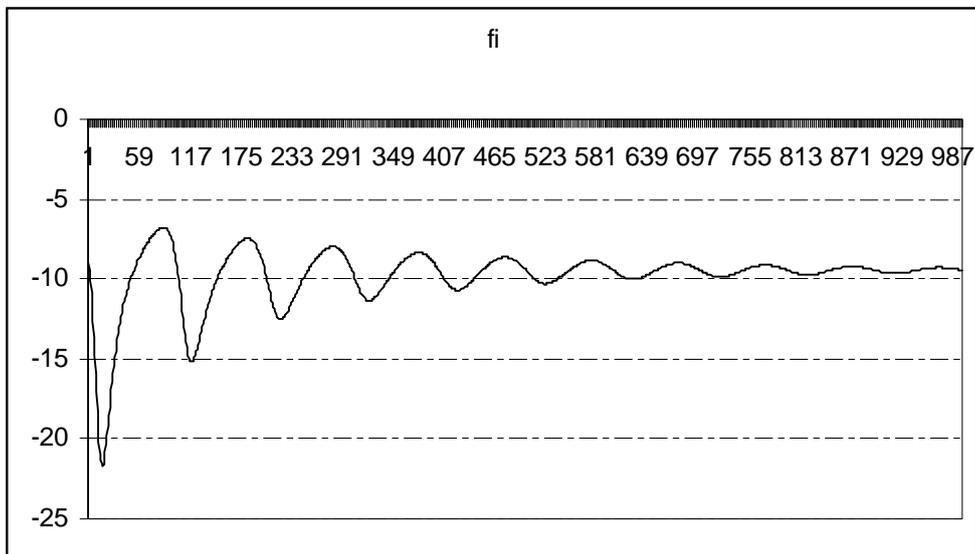
Figure 6: Numerical example: demand-output ratio dynamics.

Figure 7: Numerical example: CMI - output ratio dynamics.